Supplementary Material

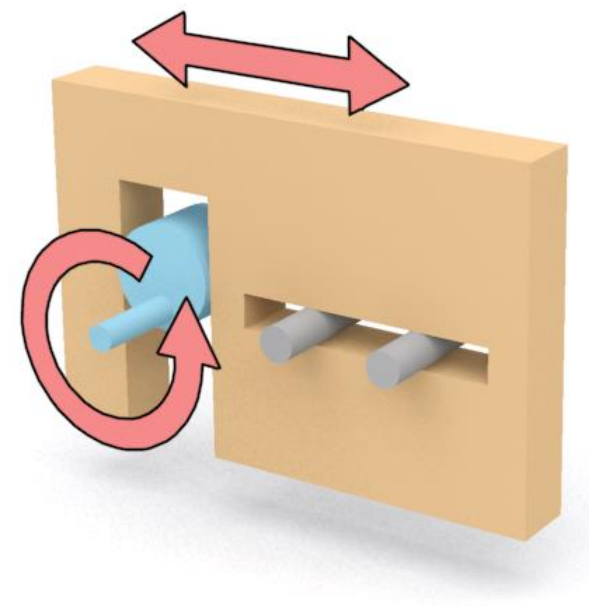
for “Computational Design of Wind-up Toys”

Contents

Part 1:
Kinematic Equations of Elemental Mechanisms

Elemental Mechanism Table

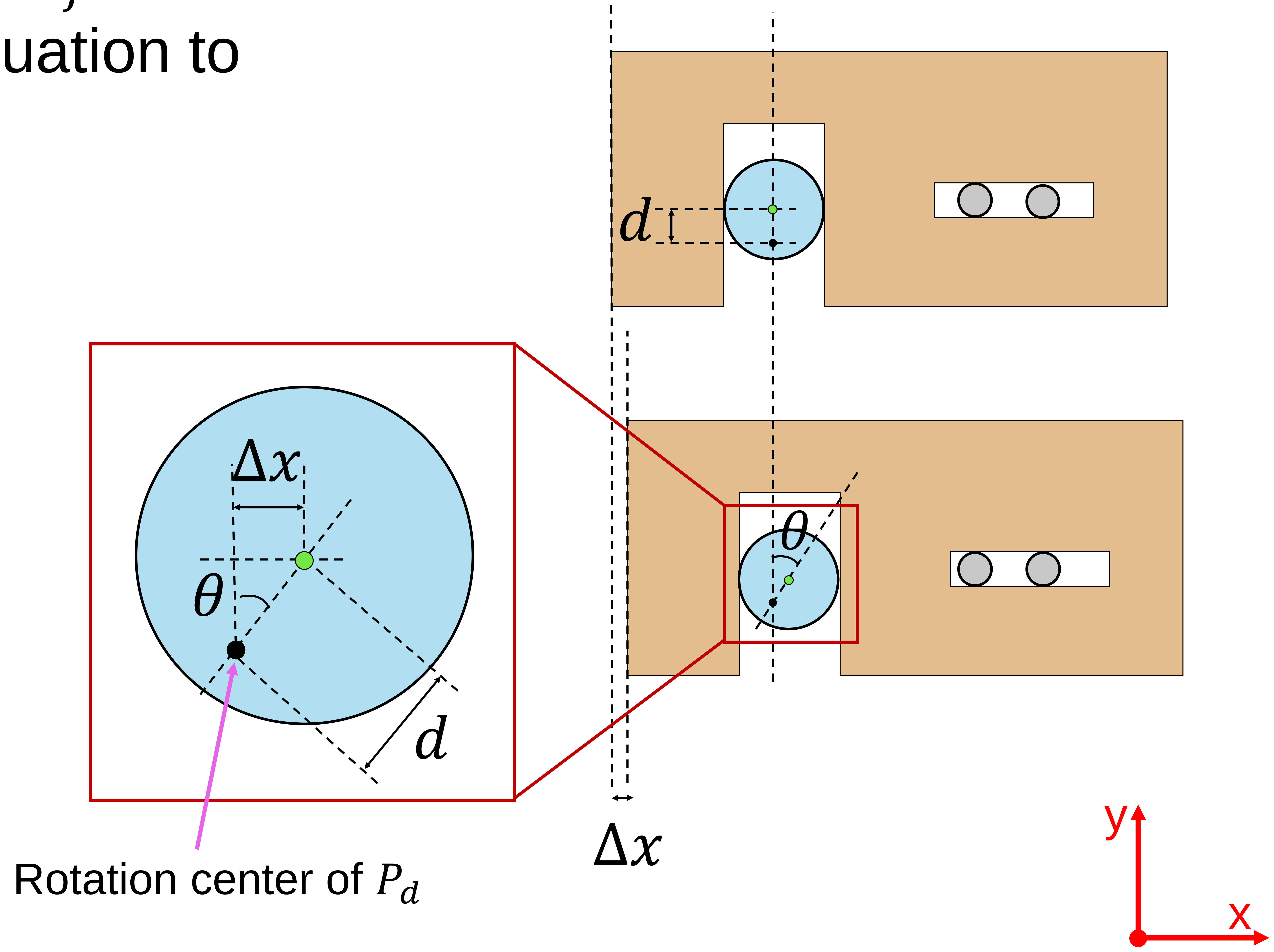
Denoting the pose of driver P_d (in blue) and follower P_f (in brown) at time t as $M_d(t)$ and $M_f(t)$, respectively, the goal of modeling an elemental mechanism's kinematics is to be able to compute $M_f(t)$ from $M_d(t)$ for all t. The following slides will formulate kinematic equations for each elemental mechanism.

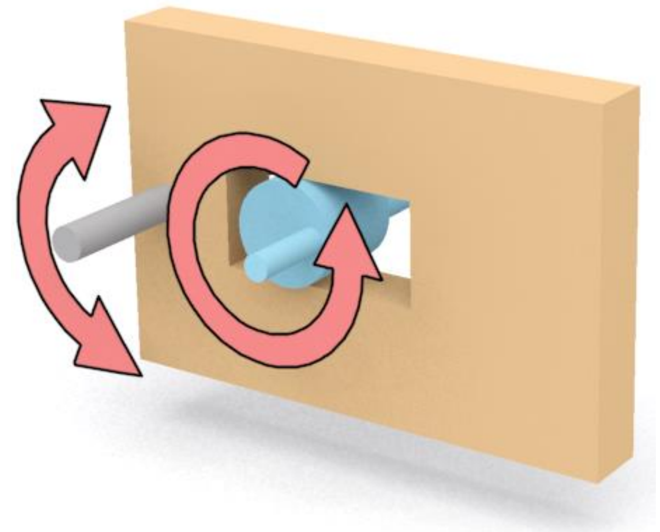


#1 $R_z \rightarrow T_x$

Denote P_d's rotation angle as θ, and P_f 's translation distance along x-axis as Δx. The equation to compute Δx is:

$$\Delta x = d \sin \theta$$





#2 $R_z \rightarrow O_z$

Denote P_d's rotation angle as θ, and P_f's rotation angle as α. The equation to compute α is:

$$\alpha = 90^{\circ} - (\alpha_1 + \alpha_2)$$

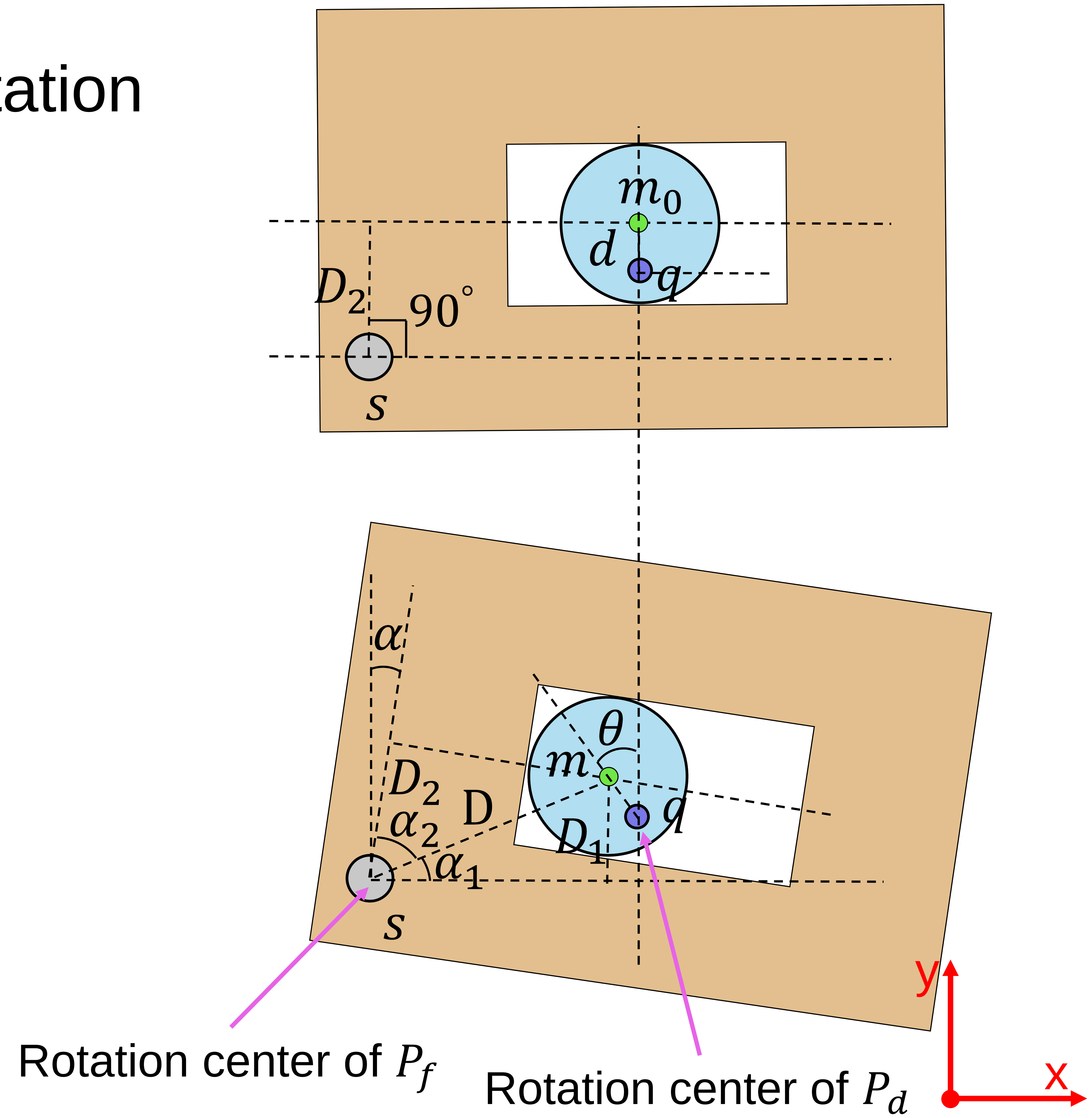
where

$$\alpha_1 = \cos^{-1}\frac{D_1}{D} \qquad \alpha_2 = \cos^{-1}\frac{D_2}{D}$$

$D = \|m - s\|$

$D_1 = |m_y - s_y|$

$D_2 = |m_{0_y} - s_y|$



m_0 is the geometric center of the round cam P_d in the initial configuration (top figure), which is aligned vertically with the fixed cam rotation center (q). Hence, we have

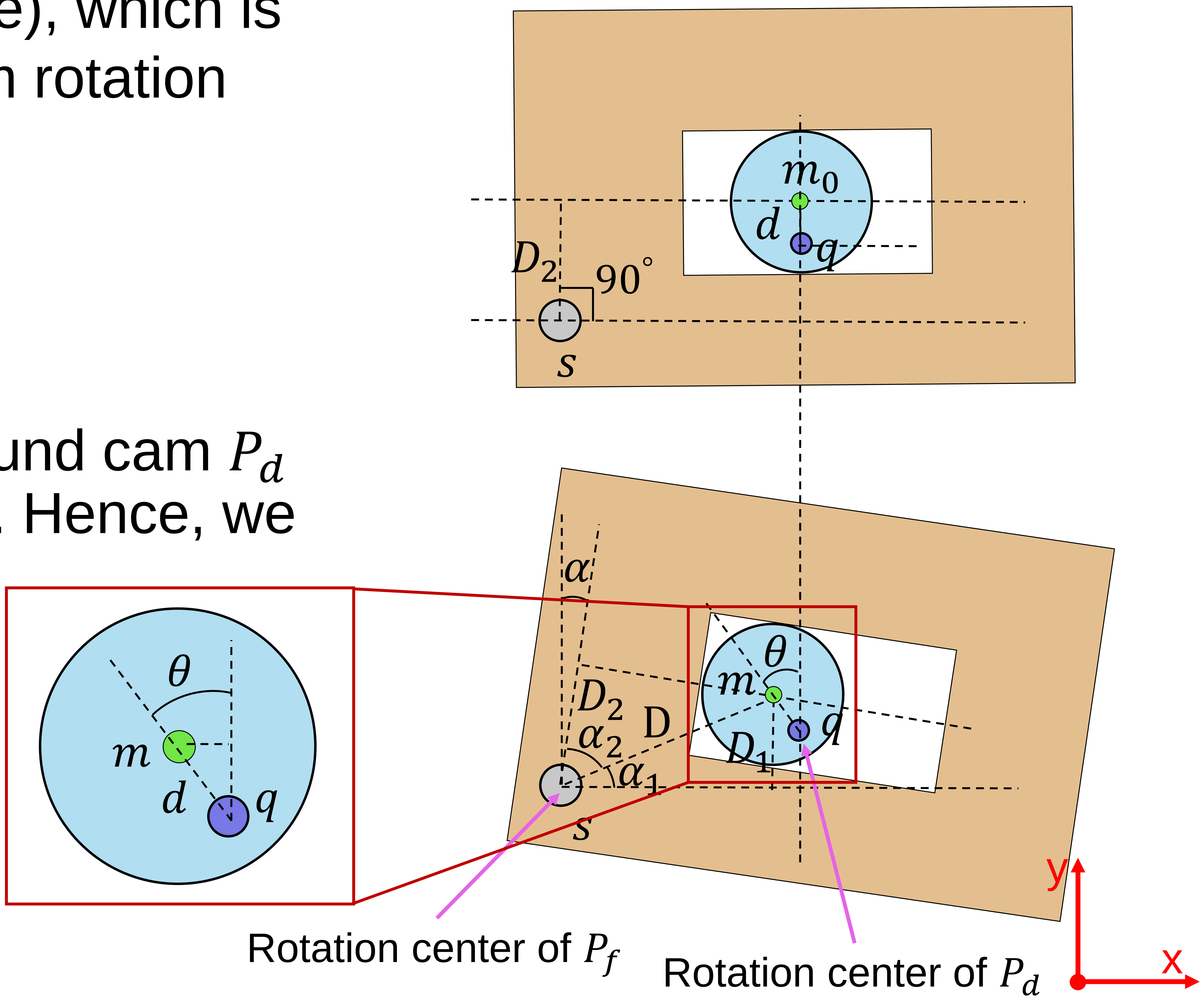
$$m_{0_x} = q_x$$

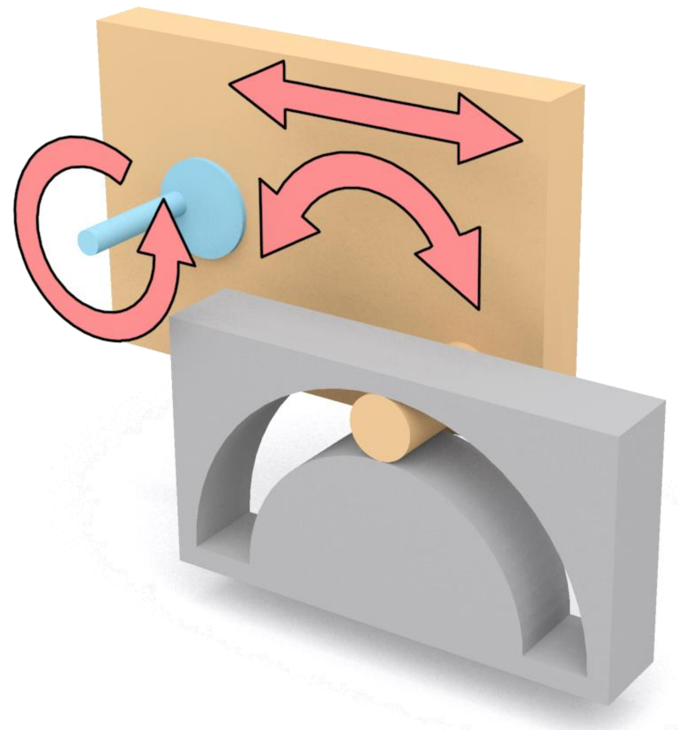
$$m_{0_y} = q_y + d$$

m is the geometric center of the round cam P_d after it rotates for θ (bottom figure). Hence, we have

$$m_x = q_x - d\sin\theta$$

$$m_y = q_y + d\cos\theta$$



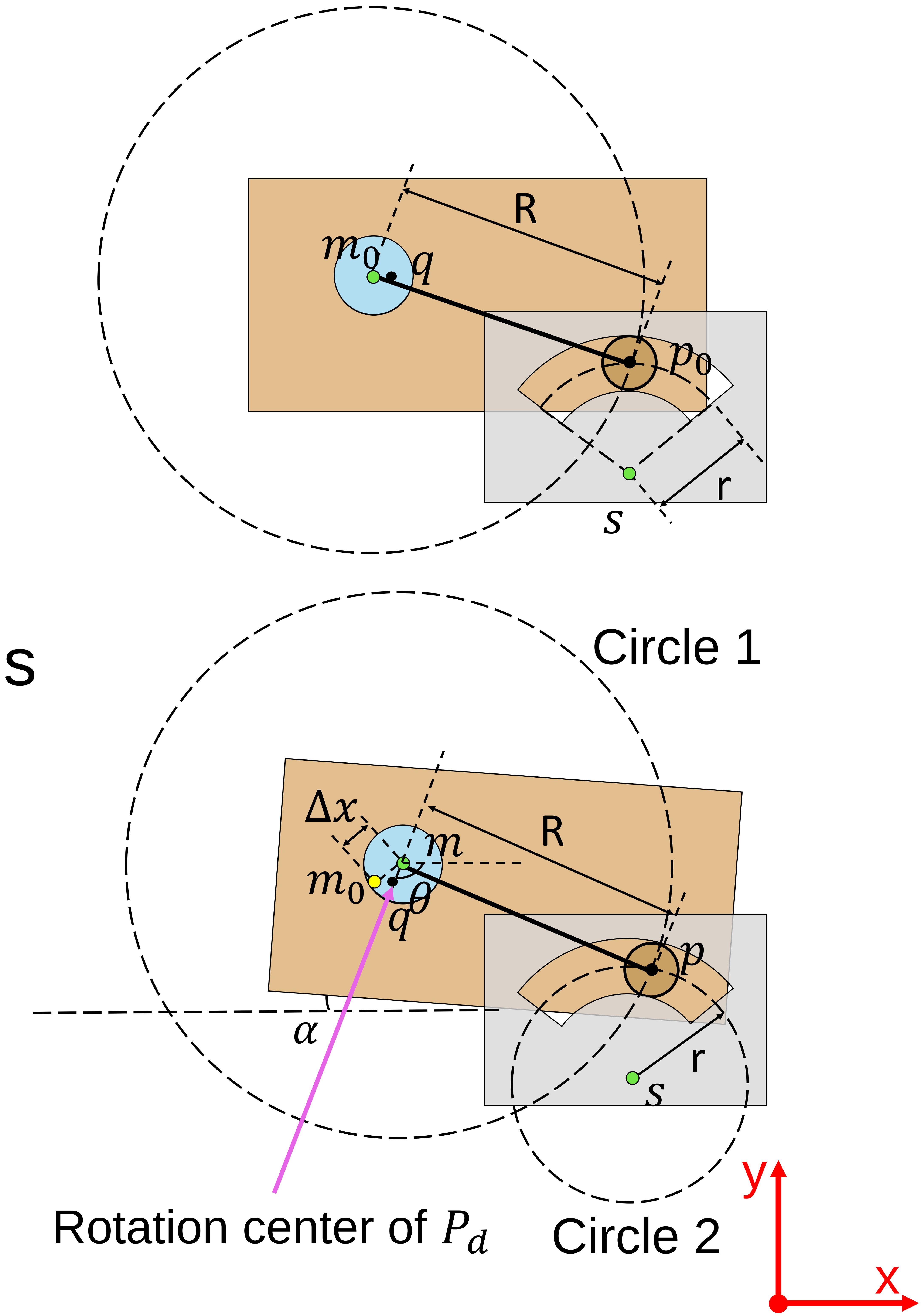


#3 $R_z \rightarrow O_z T$

Denote P_d's rotation angle as θ.

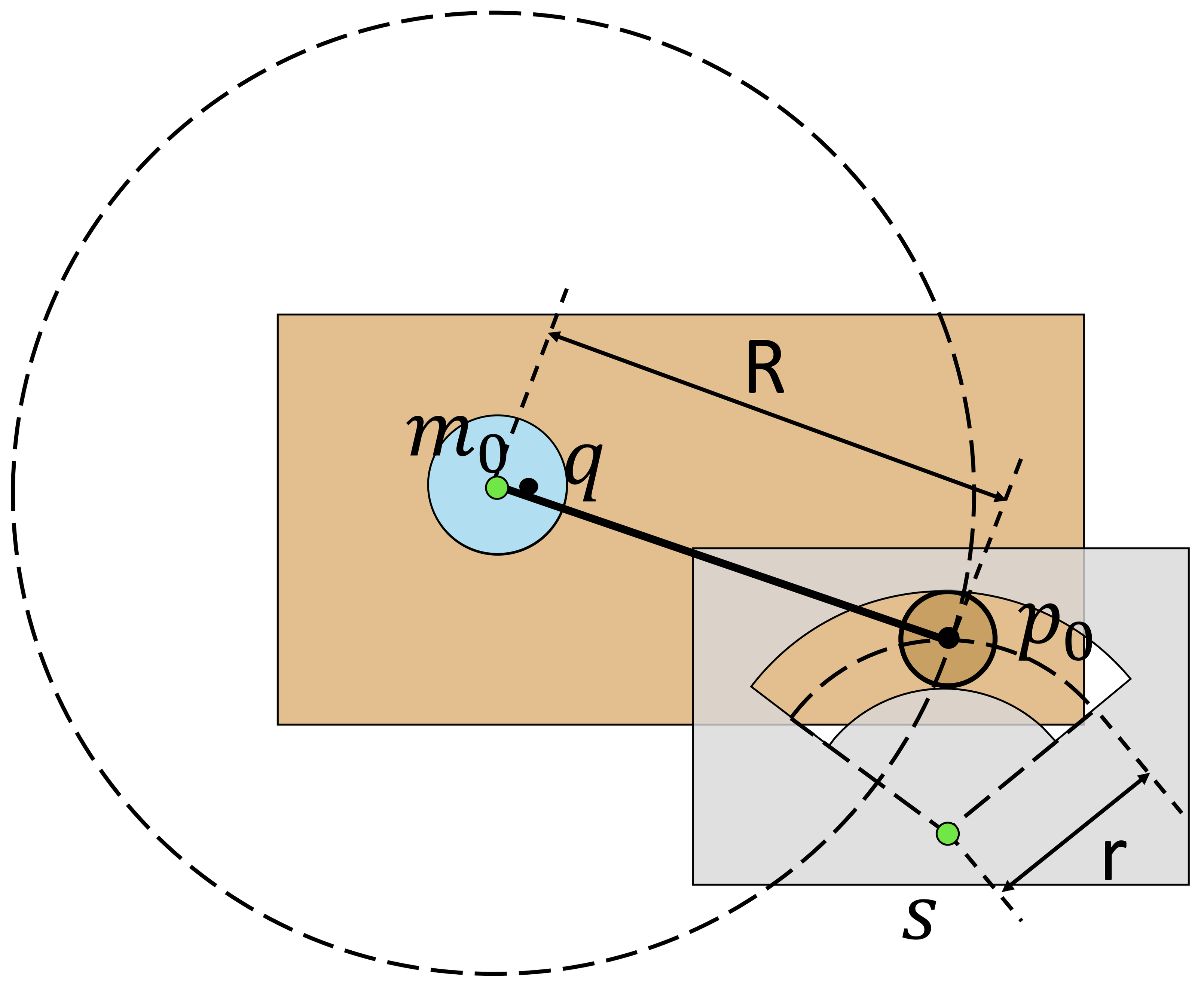
Denote P_f's rotation angle as α and translation vector as Δx.

The equation to compute α and Δt is based on computing the driver-follower joint center p, which is at the intersection point(s) between the two circles (see right figures).



The 2 circles that p locates are

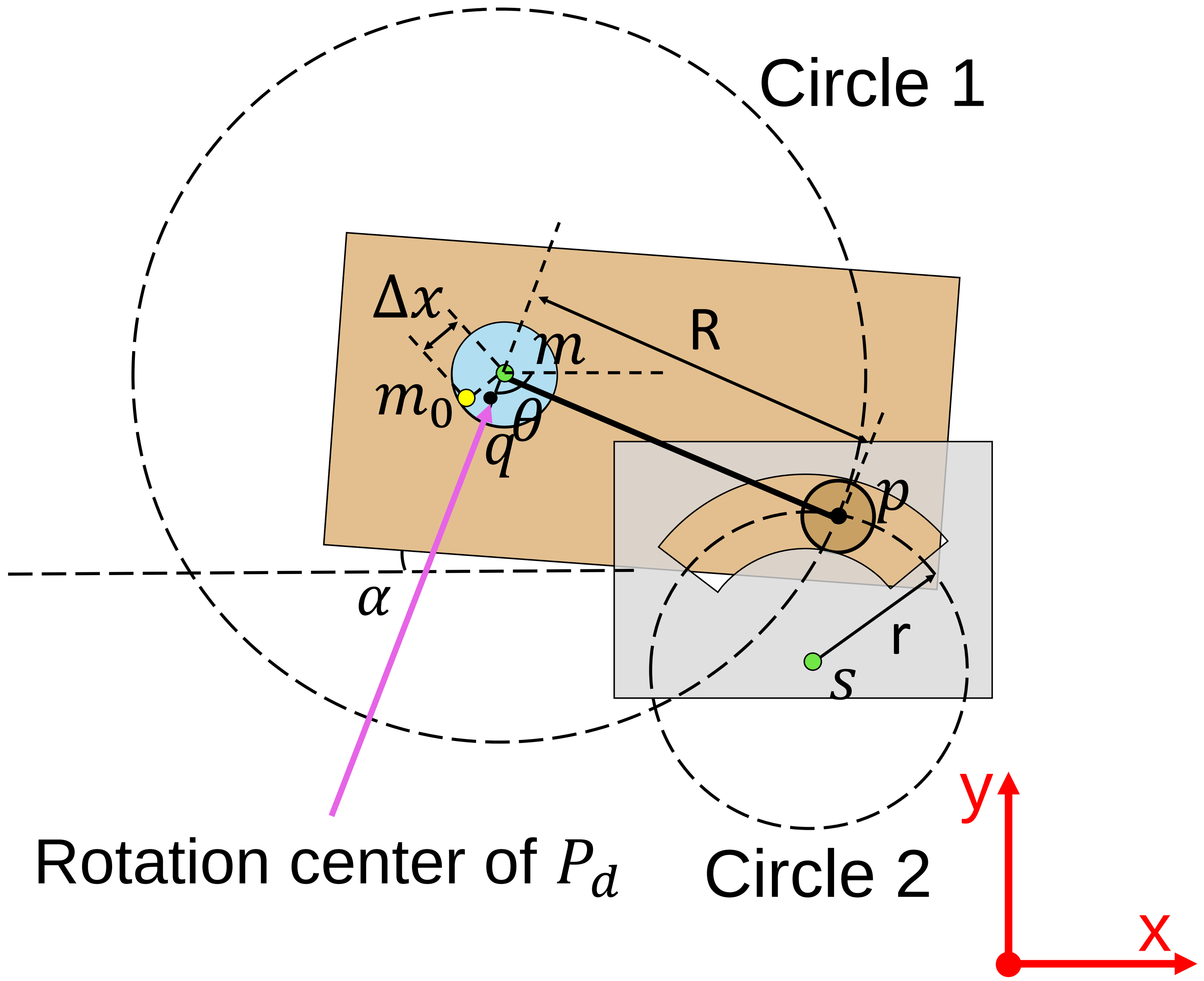
Circle 1: center m ; radius $R = ||p_0 - m_0||$
Circle 2: center s ; radius $r = ||p_0 - s||$



Hence, p must satisfy the following two equations

$$\begin{cases}(p_x - m_x)^2 + (p_y - m_y)^2 = R^2 \\ (p_x - s_x)^2 + (p_y - s_y)^2 = r^2\end{cases}$$

Note that m_0 and m are computed in the same way as in #2 $R_z \rightarrow O_z$



By solving the above two equations, we have

$$p_y = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + s_y$$

$$p_x = \frac{r^2 - R^2 + {m_x'}^2 + {m_y'}^2 - 2m_y'(p_y - s_y)}{2m_x'} + s_x$$

where

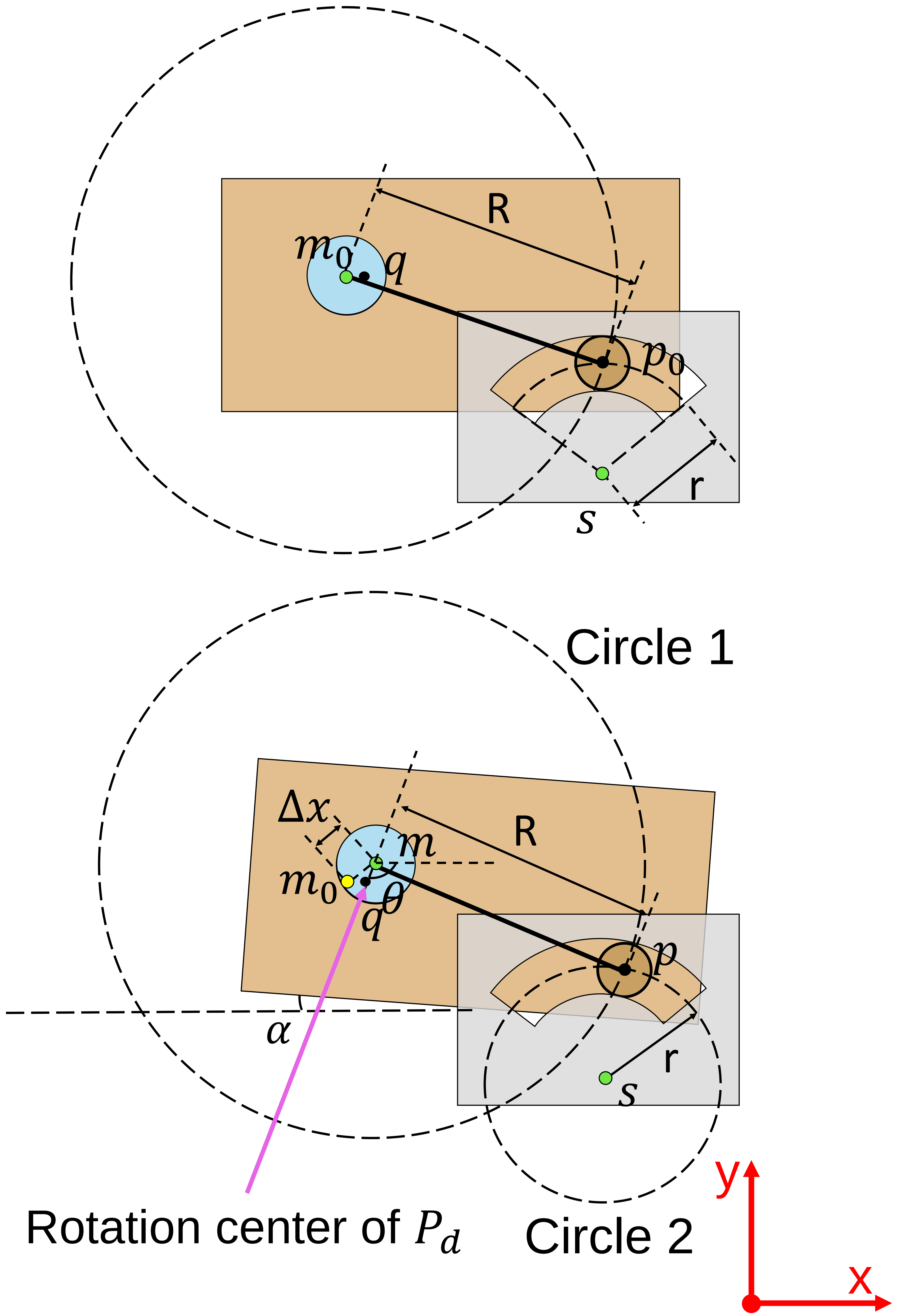
$$m_x' = m_x - s_x$$

$$m_y' = m_y - s_y$$

$$a = 4({m_x'}^2 + {m_y'}^2)$$

$$b = -4m_y'(r^2 - R^2 + {m_x'}^2 + {m_y'}^2)$$

$$c = (r^2 - R^2 + {m_x'}^2 + {m_y'}^2)^2 - (2m_x'R)^2$$



Based on the calculated p, we have

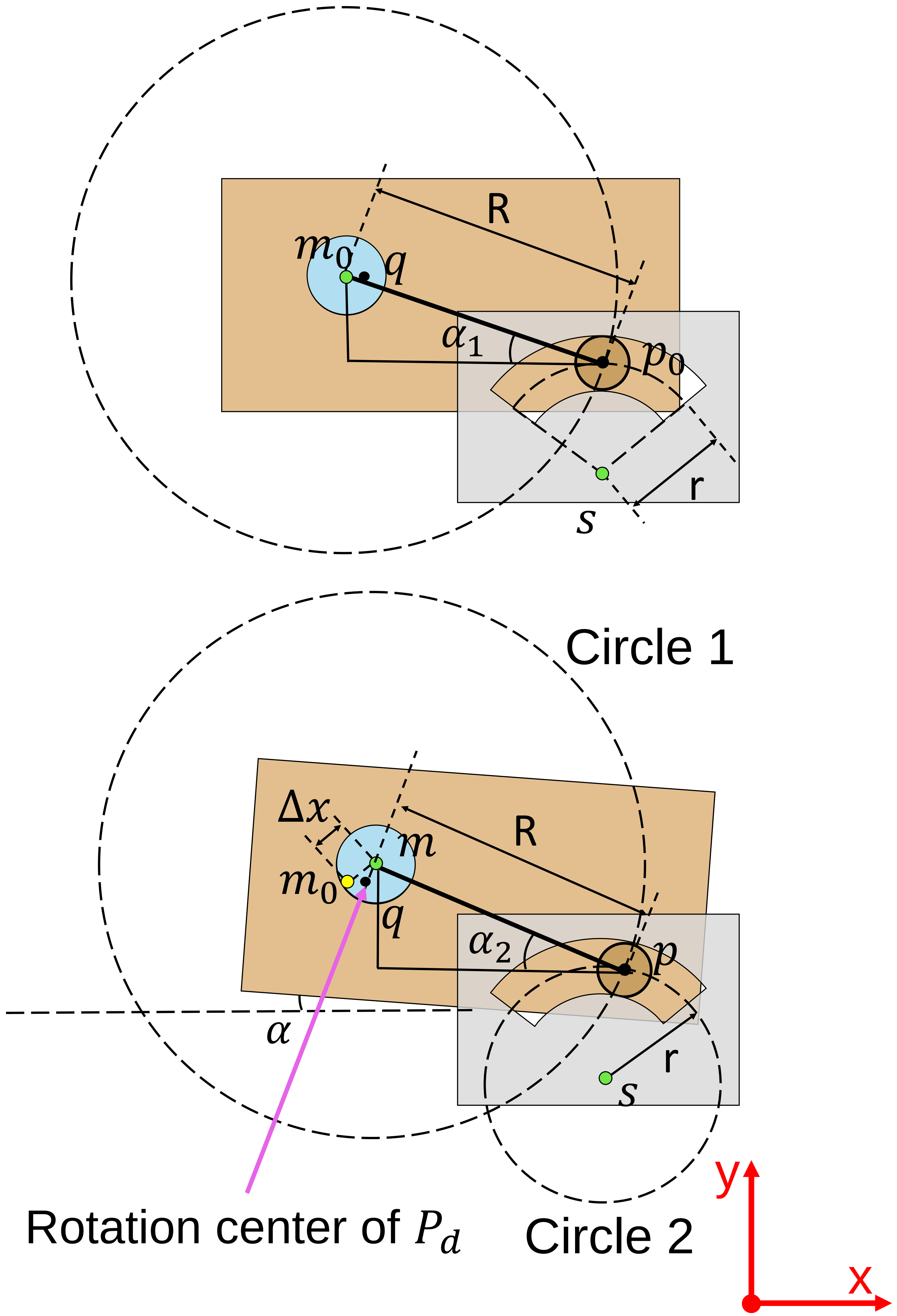
$$\Delta x = p - p_0$$

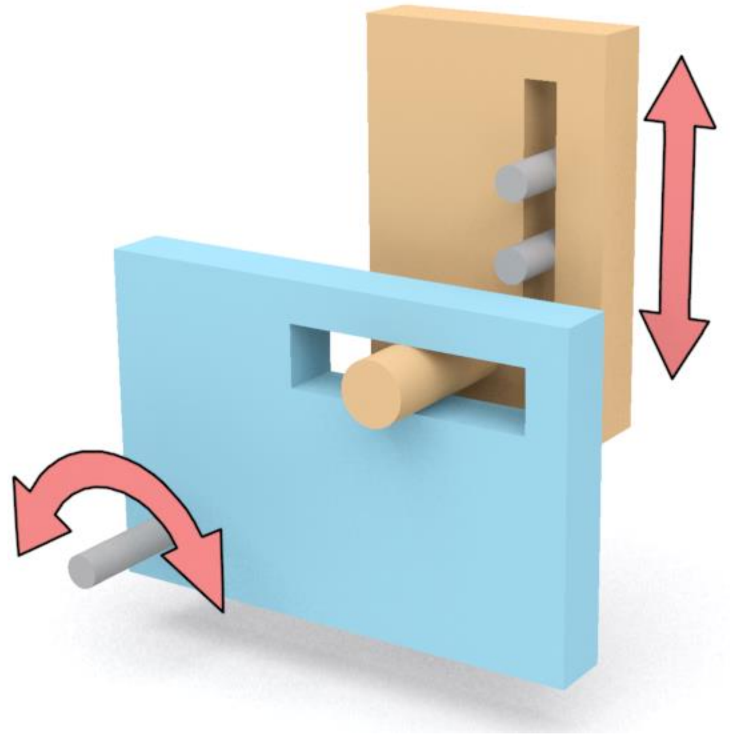
$$\alpha = \alpha_2 - \alpha_1$$

where

$$\alpha_1 = \tan^{-1}\left(\frac{p_{0_y} - m_{0_y}}{p_{0_x} - m_{0_x}}\right)$$

$$\alpha_2 = \tan^{-1}\left(\frac{p_y - m_y}{p_x - m_x}\right)$$





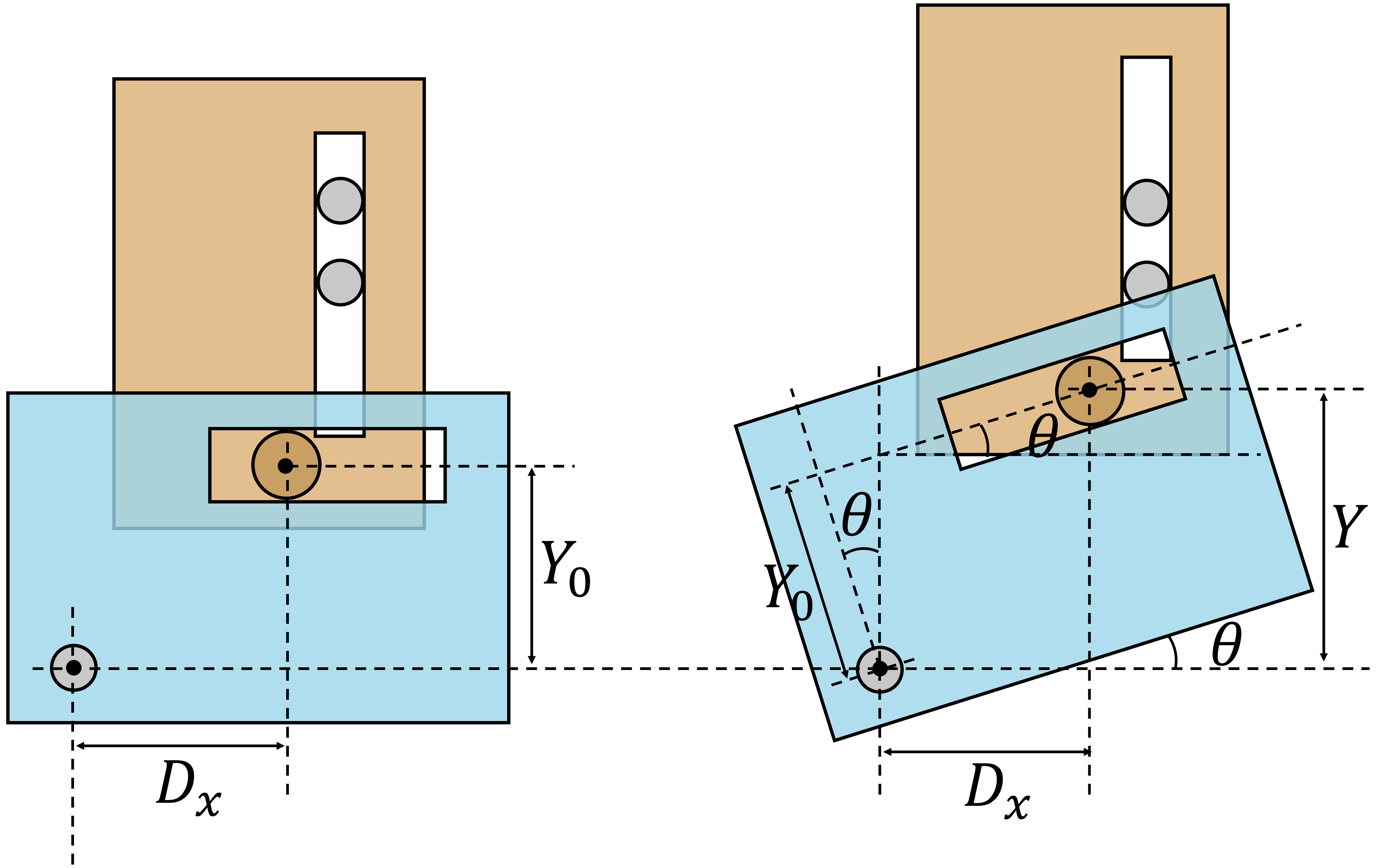
#4 $O_z \rightarrow T_y$

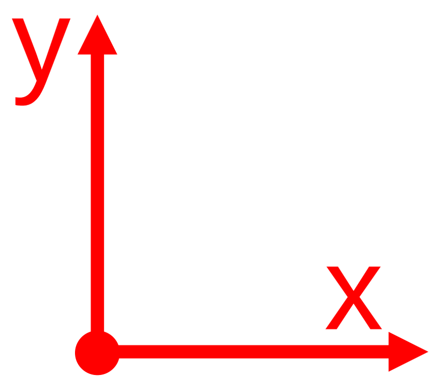
Denote P_d's rotation angle as θ, and P_f 's translation distance along y-axis as Δy. The equation to compute Δy is:

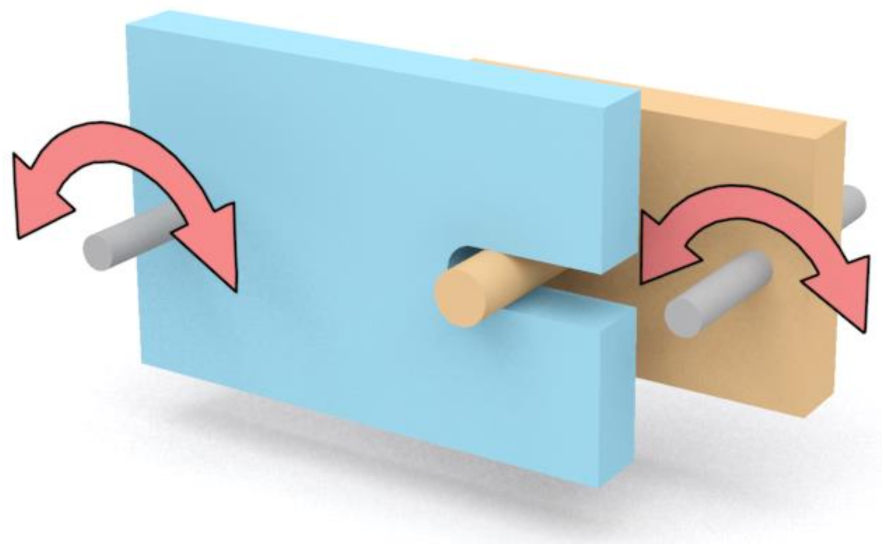
$$\Delta y = Y - Y_0$$

where

$$Y = \frac{Y_0}{\cos\theta} + D_x \tan\theta$$







#5 $O_z \rightarrow O_z$

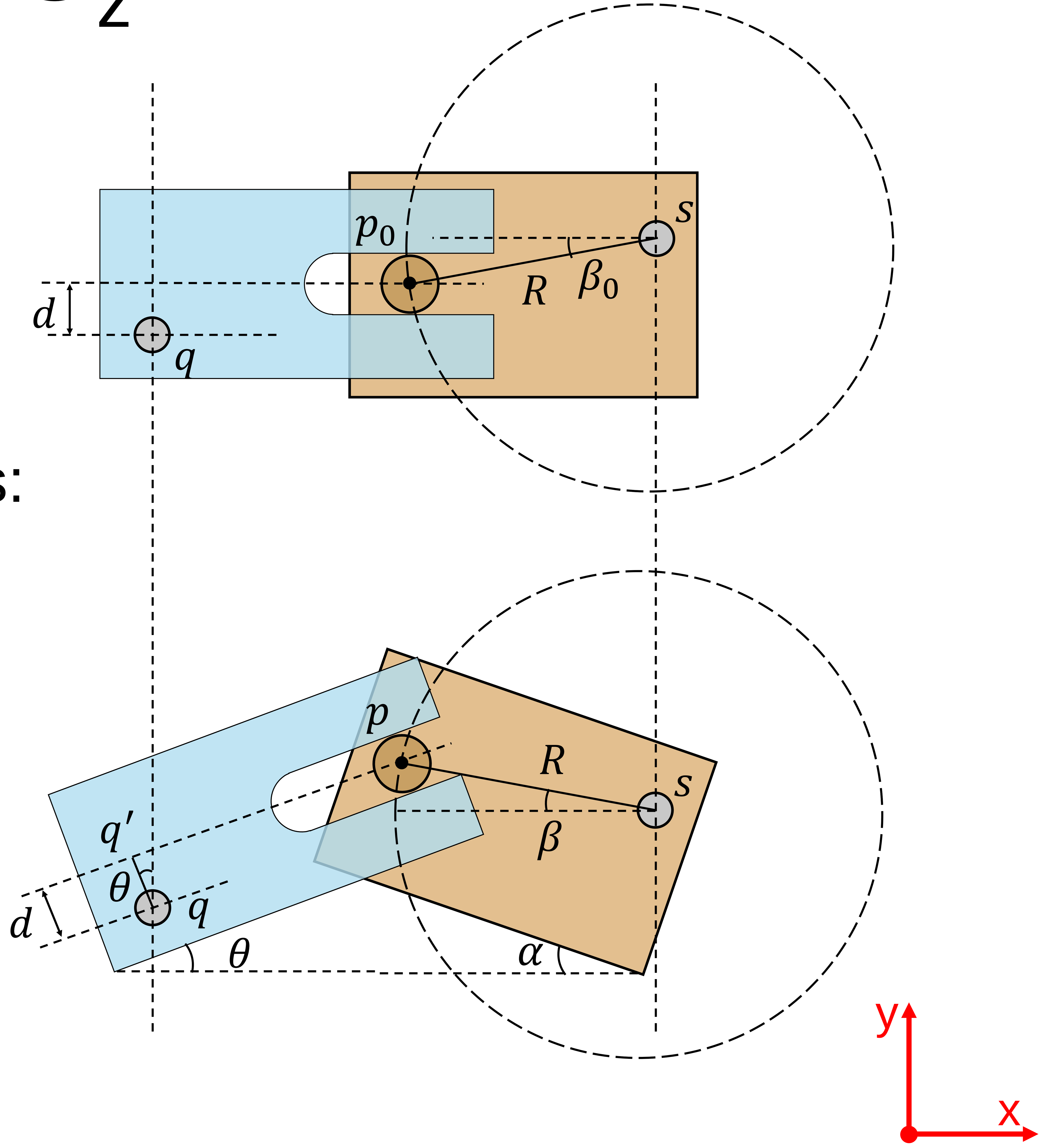
Denote P_d's rotation angle as θ, and P_f's rotation angle as α. The equation to compute α is based on computing the driver-follower joint center p, which can be computed using the following two equations:

$$\begin{cases} (p_x - s_x)^2 + (p_y - s_y)^2 = R^2 \\ \dfrac{p_y - q'_y}{p_x - q'_x} = \tan\theta \end{cases}$$

where

$$q'_x = q_x - d \sin\theta$$

$$q'_y = q_y + d \cos\theta$$



By solving the above two equations, we get

$$p_x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

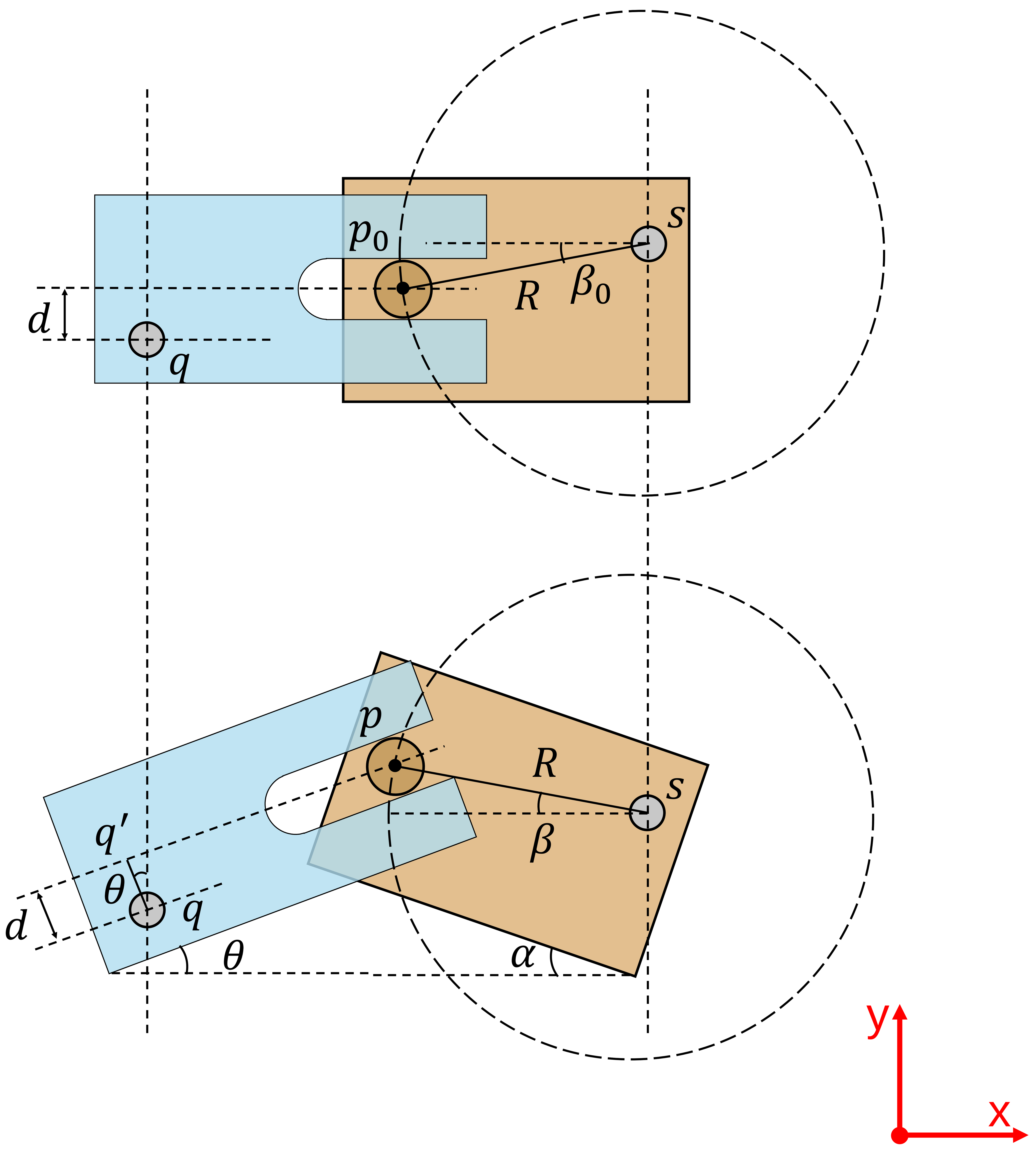
$$p_y = \tan(p_x - q'_x) + q'_y$$

where

$$a = 1 + (\tan\theta)^2$$

$$b = -2\, s_x - 2\left(-q'_x \tan\theta \ + q'_y \ \right) - 2\, s_y\, tan\theta$$

$$c = {s_x}^2 + (-\, q'_x \tan\theta + q'_y)^2$$
$$-\ \ 2\, s_y\left(-\, q'_x \tan\theta + q'_y\right) + {s_y}^2$$



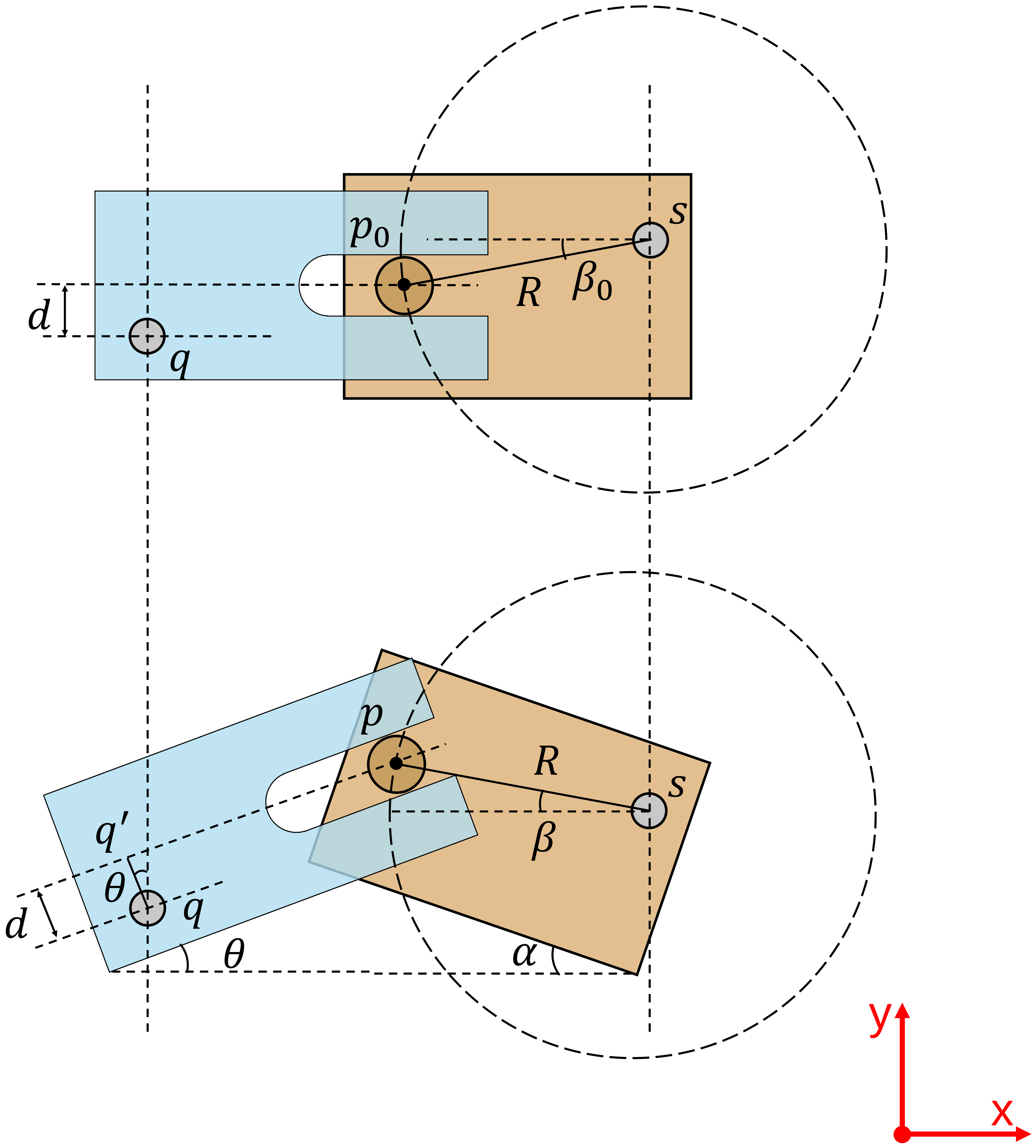
Based on the calculated p, we have

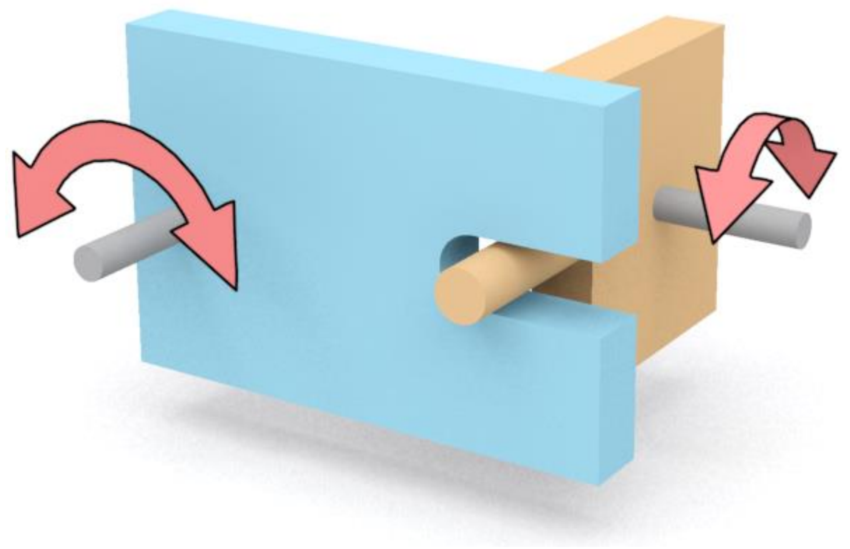
$$\alpha = \beta_0 + \beta$$

where

$$\beta_0 = \tan^{-1}\frac{s_y - p_{0_y}}{s_x - p_{0_x}}$$

$$\beta = \tan^{-1}\frac{p_y - s_y}{s_x - p_x}$$



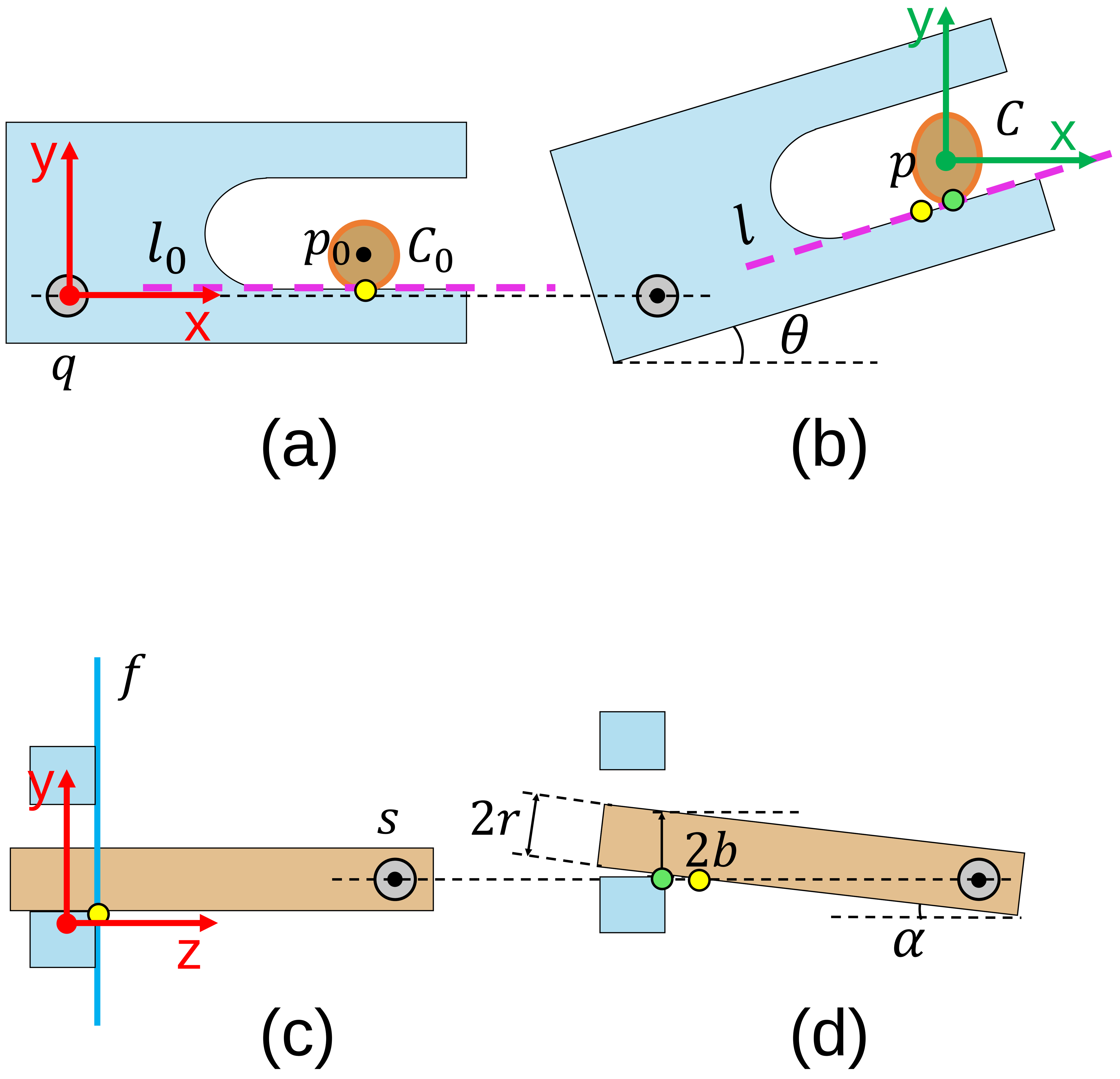


#6 $O_z \rightarrow O_x$

Denote P_d's rotation angle as θ, and P_f's rotation angle as α.

The equation to compute α is based on the contact between a line l (in purple) in driver's major plane f, and the projection of driver-follower joint on f, which is actually an oval, denoted as C (in orange), see (a&b).

Line l and oval C should always contact each other during the parts motion. The initial contact point is colored in yellow while the current contact point is colored in green.



Denote the center of C as p, we build two coordinate systems:
1) red one centered at q in (a); and 2) green one centered at p in (b).
The following calculations will be done in these two coordinate systems.

Denote line l's equation in the red coordinate system as:

$$y = kx + c \qquad (1)$$

where

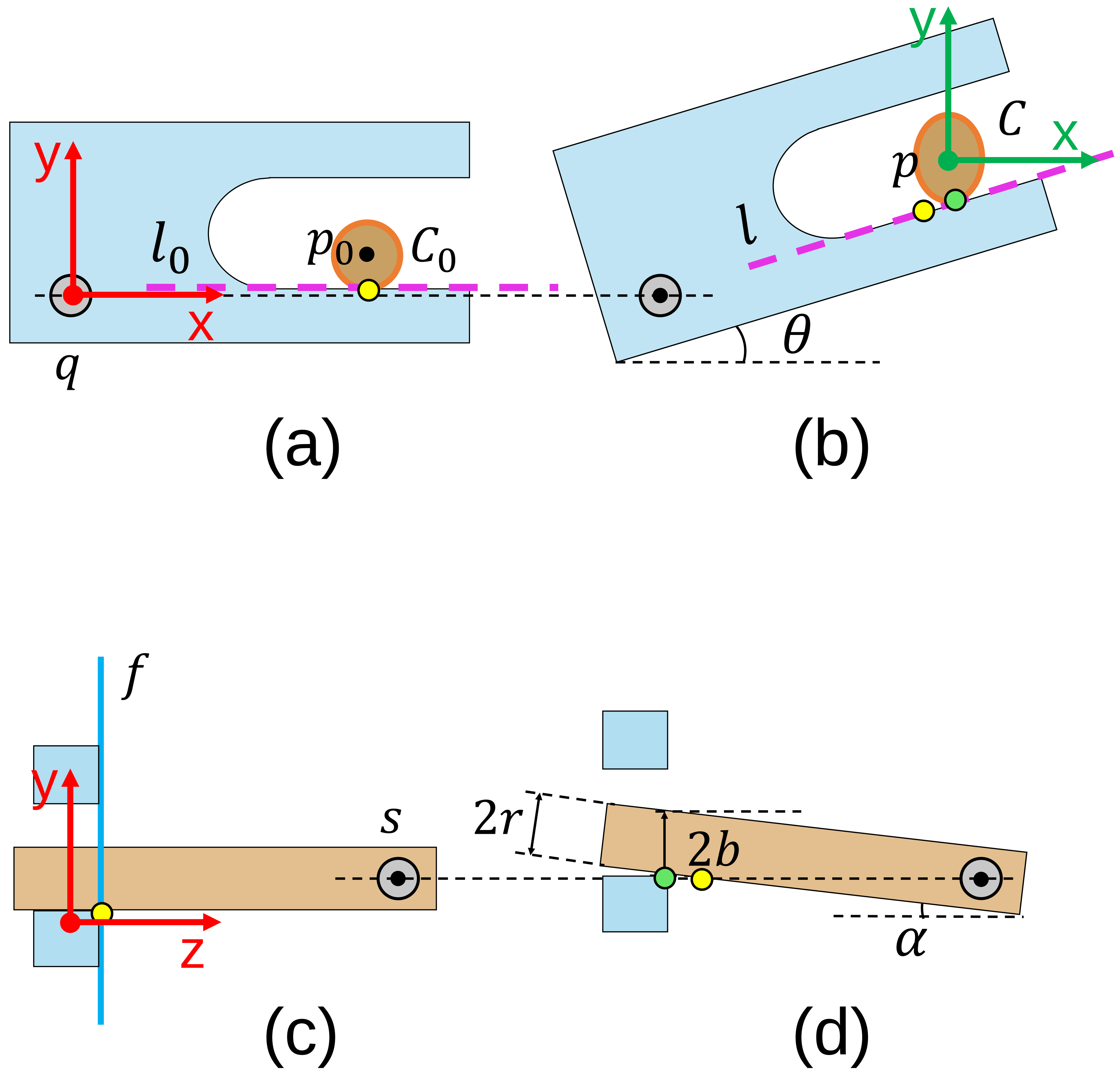
$$k = \tan\theta \qquad c = \frac{p_{0_y} - q_y - r}{\cos\theta}$$

Denote oval C's equation in the green coordinate system as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$a = r \qquad b = \frac{r}{\cos\alpha}$$



Denote line l's equation in the green coordinate system as: $y = kx + c'$

Consider the two equations together:

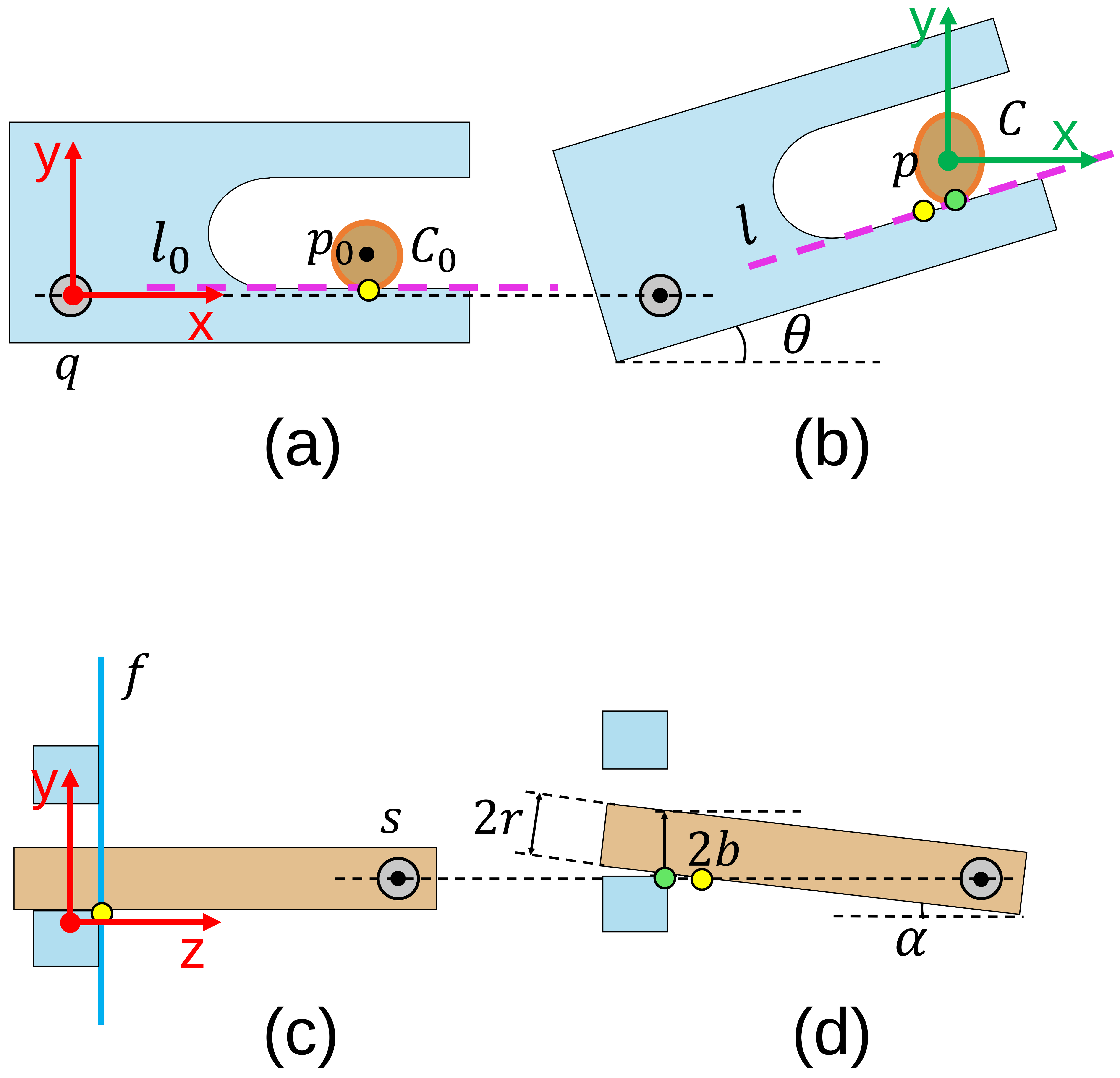
$$\begin{cases} y = kx + c' \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases}$$

Since l and C are always tangent to each other, the equation set above can have a unique solution. Hence, we have equation (2)

$$a^2k^2 - c'^2 + b^2 = 0 \qquad (2)$$

In addition, absolute coordinate of p is

$$p_x = p_{0_x}$$
$$p_y = p_{0_y} + (p_{0_z} - q_z)\tan\alpha \qquad (3)$$



Denote $L = a^2 k^2$ $M = k(p_{0_x} - q_x) + c + q_y - p_{0_y}$ $N = p_{0_z} - q_z$

So, from (1) and (3), we have $c' = M - N \tan\alpha$

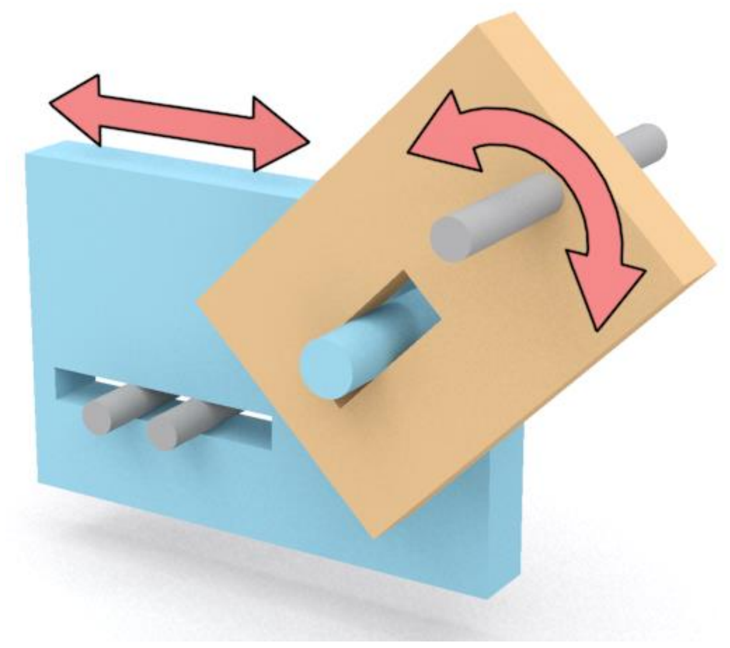
Substituting into (2), we have: $L - (M - N\tan\alpha)^2 + \frac{r^2}{cos\,\alpha^2} = 0$

Denote $\theta' = -\tan^{-1}\frac{M^2 - L - N^2}{2MN}$

We have: $r^2 - \frac{M^2 - L - N^2}{2} = sin(2\alpha + \theta')\sqrt{(\frac{M^2 - L - N^2}{2})^2 + (MN)^2}$

Hence,

$$\alpha = \frac{sin^{-1}\left(\frac{r^2 - \frac{M^2 - L - N^2}{2}}{\sqrt{\left(\frac{M^2 - L - N^2}{2}\right)^2 + (MN)^2}}\right) - \theta'}{2}$$



#7 $T_x \rightarrow O_z$

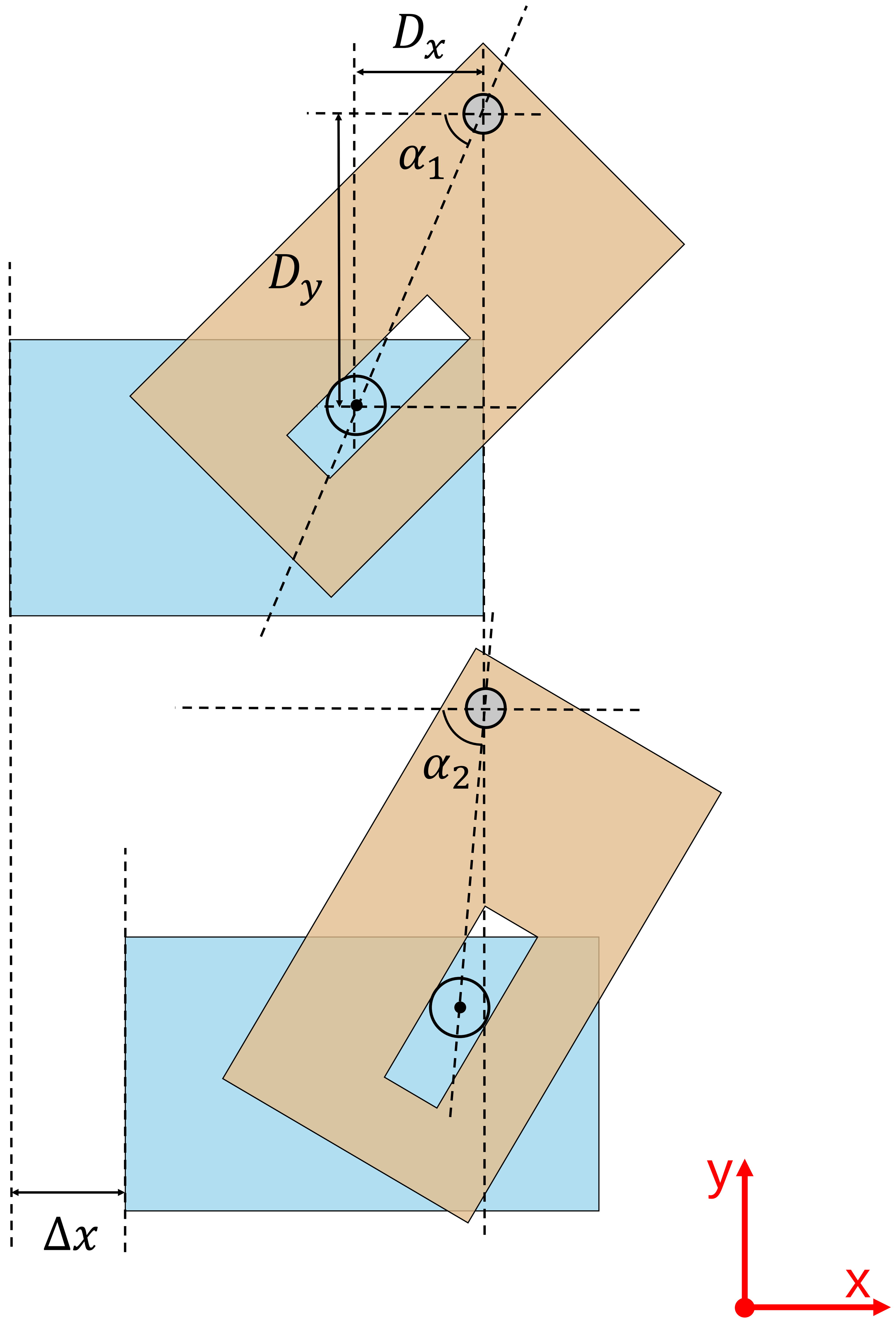
Denote P_d's translation distance along x-axis as Δx, and P_f 's rotation angle as α. The equation to compute α is:

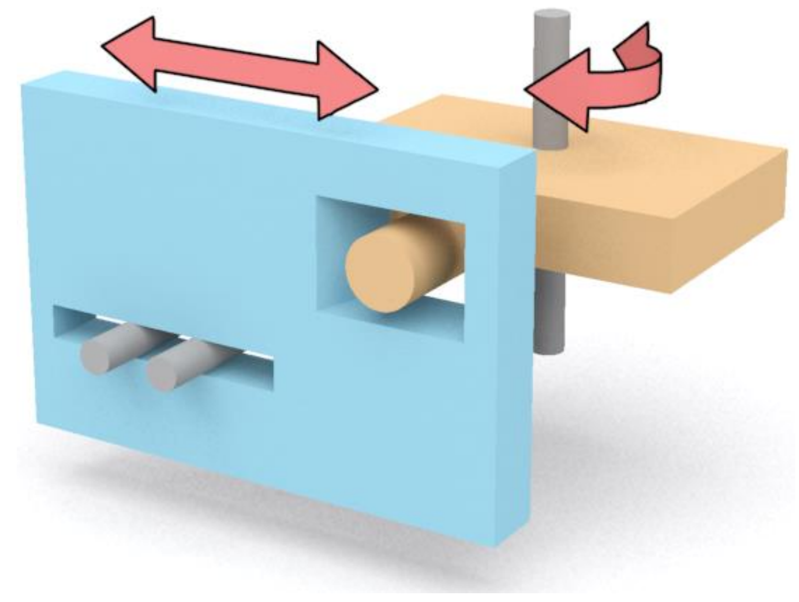
$$\alpha = \alpha_2 - \alpha_1$$

where

$$\alpha_1 = \tan^{-1}\frac{D_y}{D_x}$$

$$\alpha_2 = \tan^{-1}\frac{D_y}{D_x - \Delta x}$$





#8 $T_x \rightarrow O_y$

Denote Δx as P_d's translation distance along x-axis. We need to compute P_f's oscillating angle α around y-axis centered at point B.

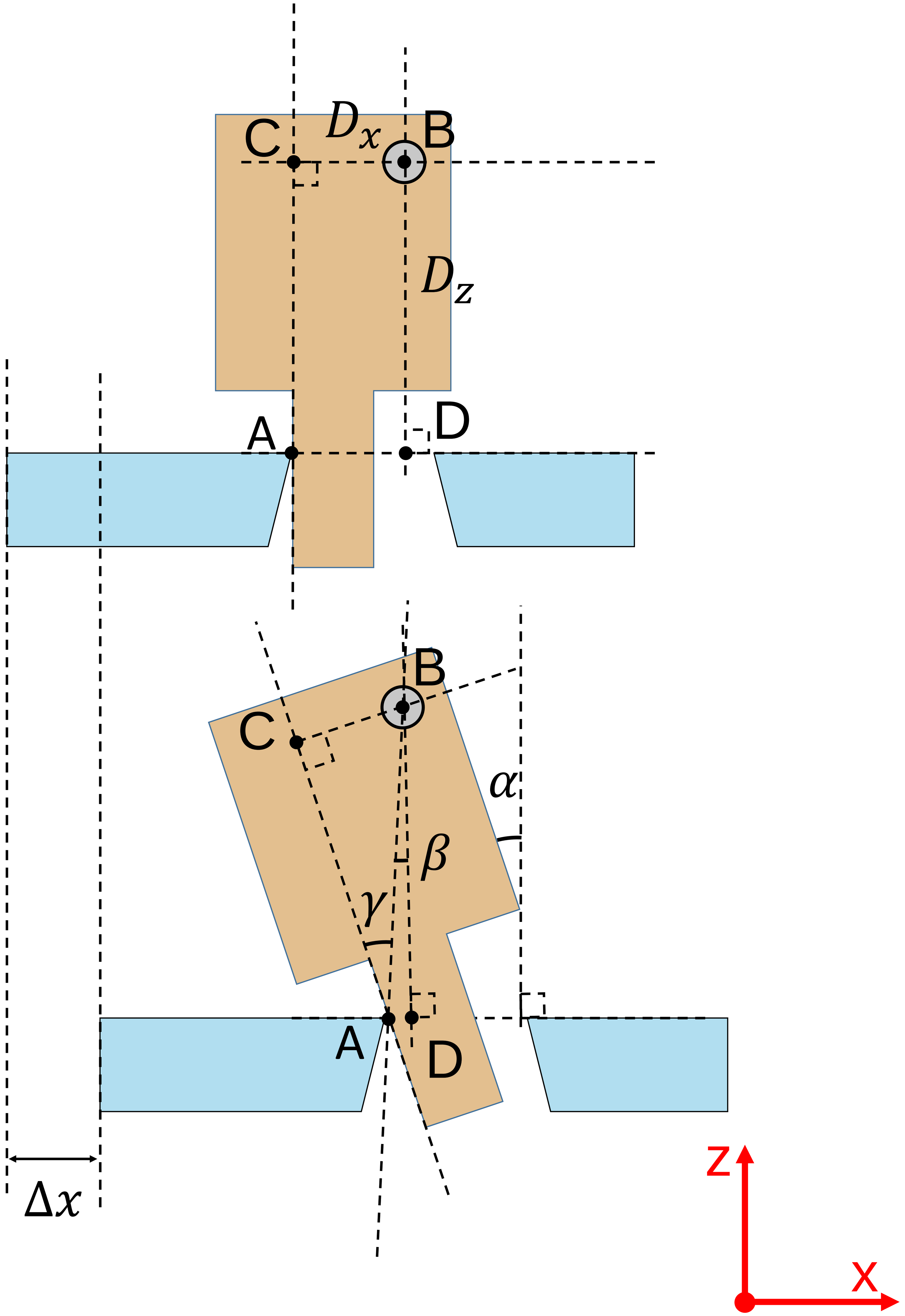
Here we assume P_d and P_f always contact at point A, e.g., due to gravity of P_f or external forces on P_f

Denote:

$$D_x = |BC| \qquad D_z = |BD|$$

$$\gamma = \angle BAC \qquad \beta = \angle ABD$$

$$A = (A_x, A_y) \quad B = (B_x, B_y)$$



From the right figure, we can see

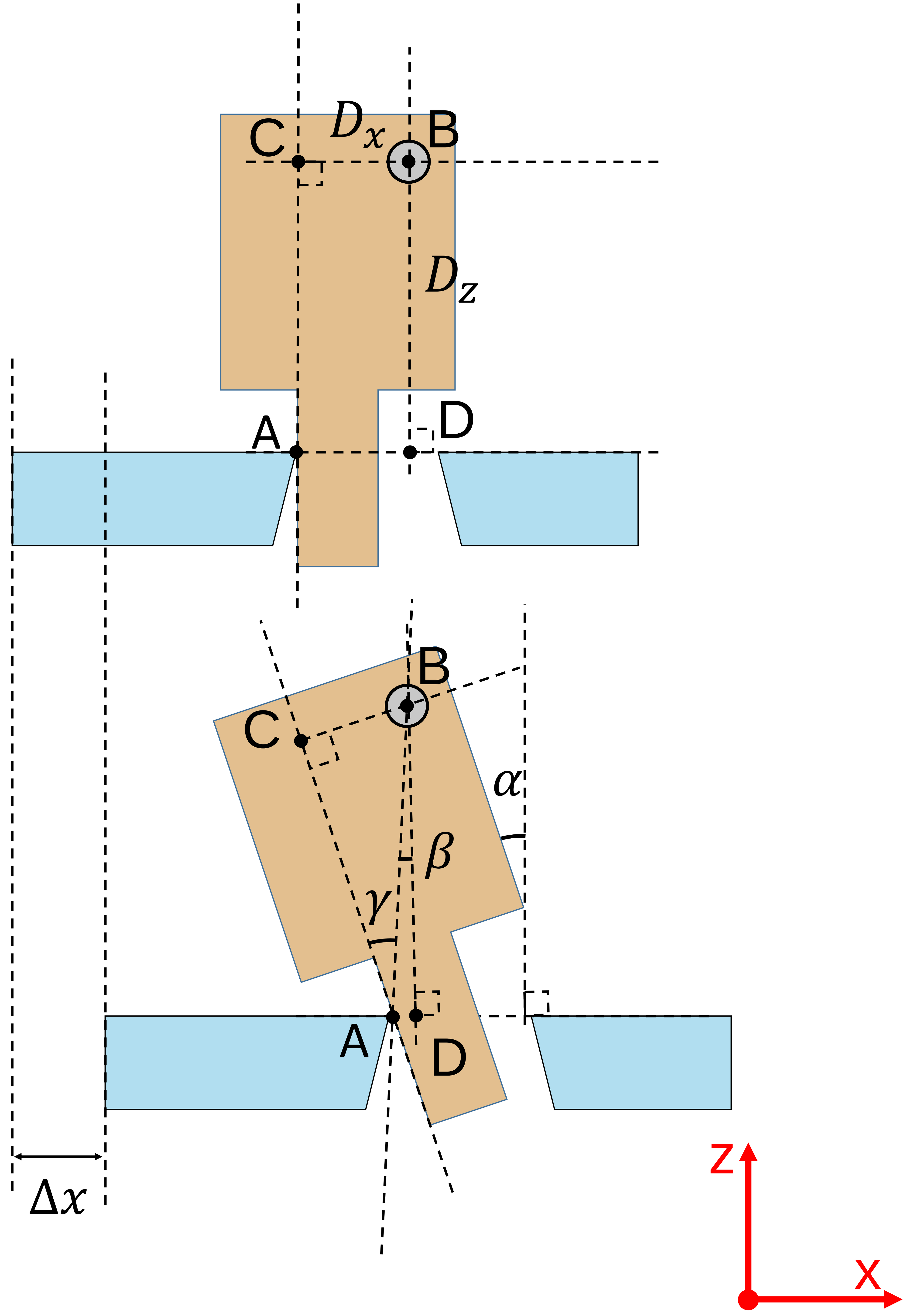
$$\alpha = \gamma - \beta$$

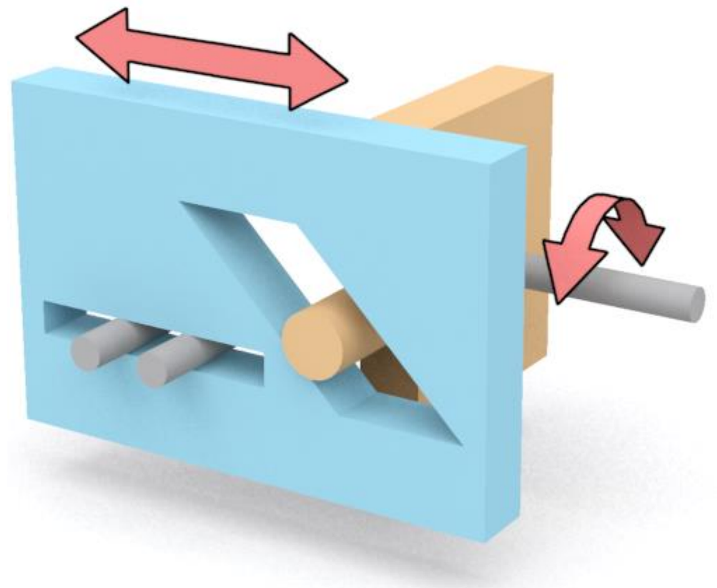
where

$$\gamma = \sin^{-1}\frac{D_x}{|BA|}$$

$$\beta = \cos^{-1}\frac{D_z}{|BA|}$$

$$|BA| = \sqrt[2]{(B_x - A_x - \Delta x)^2 + \left(B_y - A_y\right)^2}$$



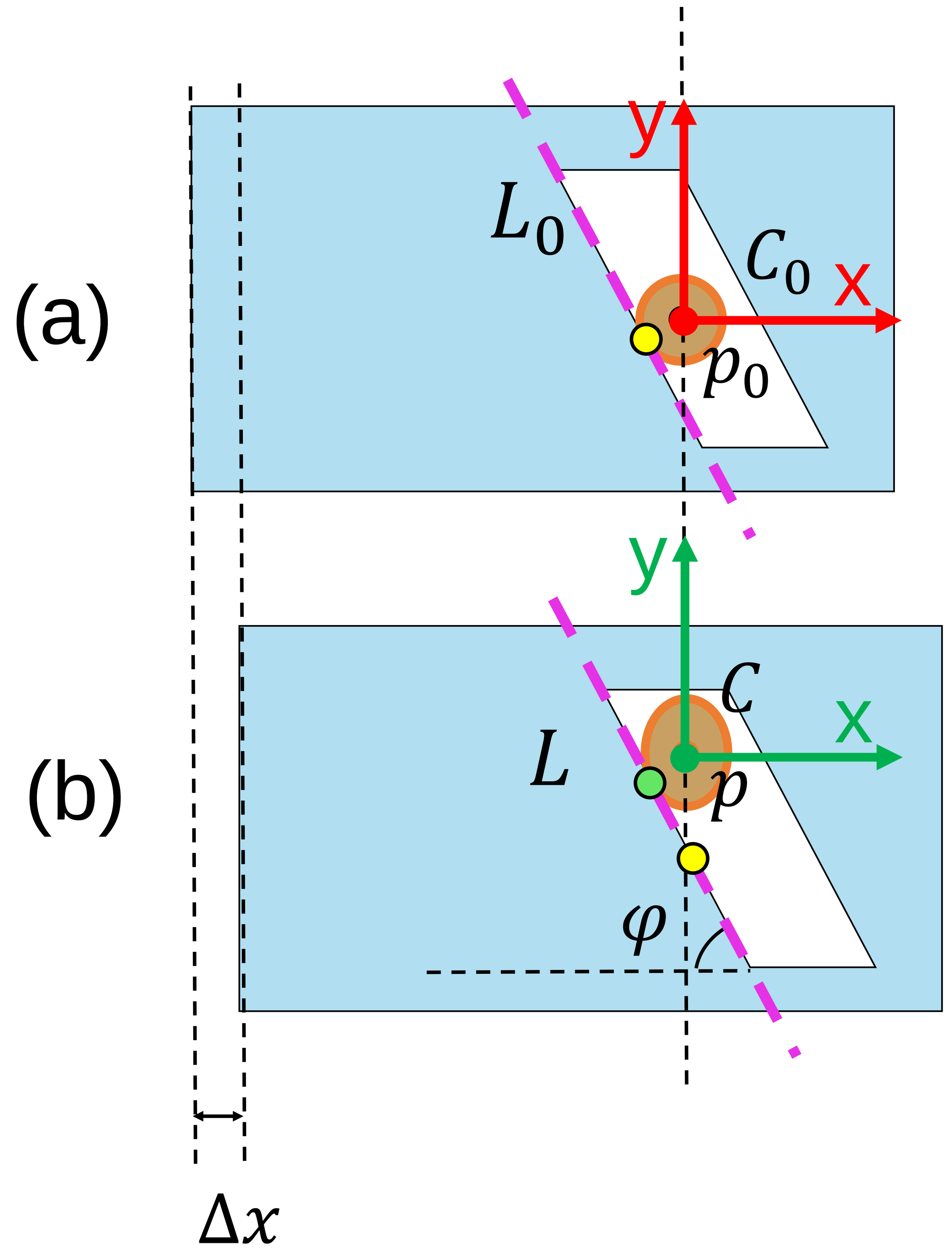


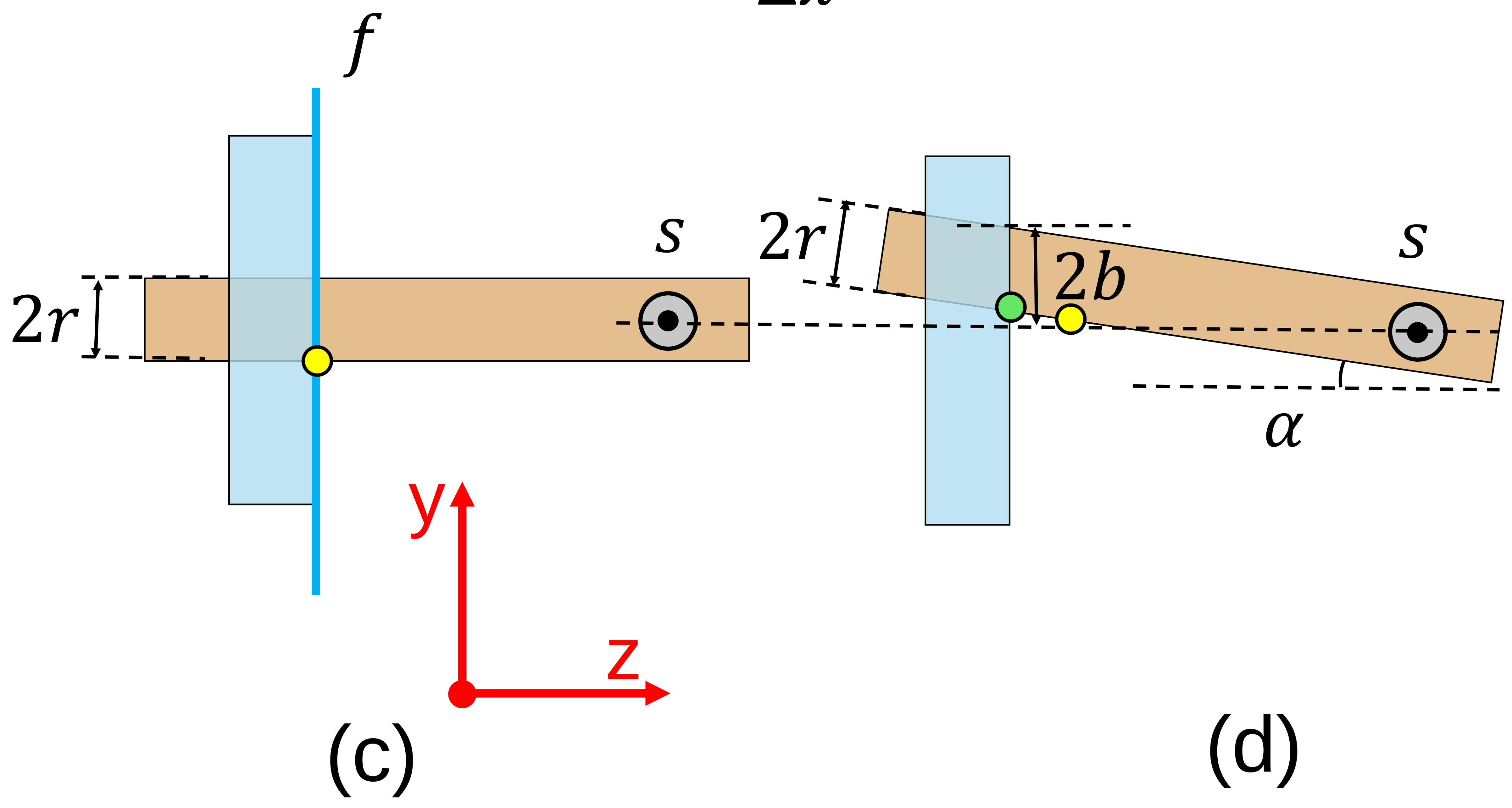
#9 $T_x \rightarrow O_x$

Denote P_d's translation distance along x-axis as Δx, and P_f's rotation angle as α.

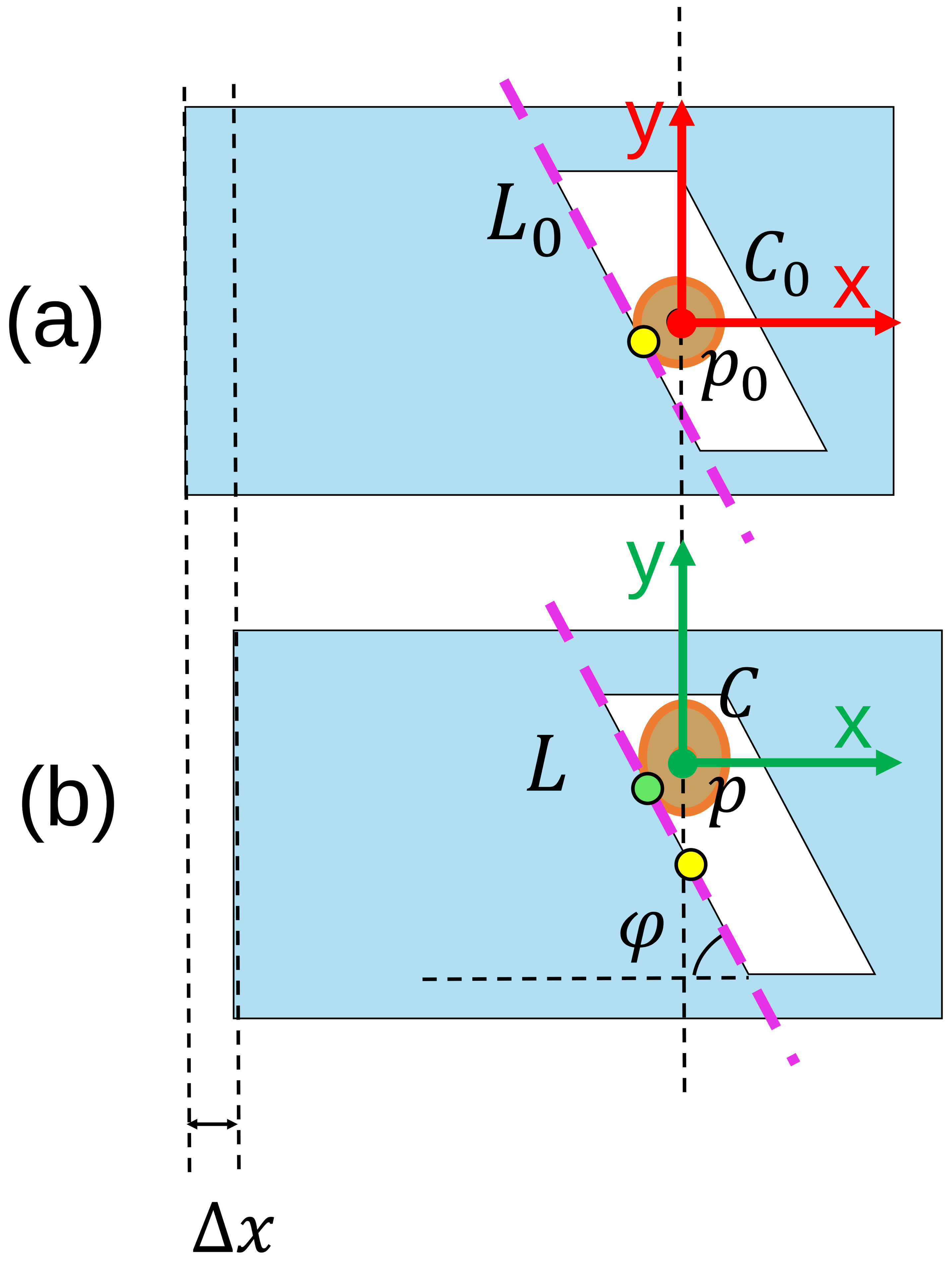
The equation to compute α is based on the contact between a line L (in purple) in driver's major plane f, and the projection of driver-follower joint on f, which is actually an oval, denoted as C (in orange), see (a&b).

Line L and oval C should always contact each other during the parts motion. The initial contact point is colored in yellow while the current contact point is colored in green.





Denote the center of C as p, we build two coordinate systems: 1) red one centered at p_0 in (a); and 2) green one centered at p in (b). The following calculations will be done in these two coordinate systems.



Denote line L's equation in the red coordinate system as:

$$y = kx + c \qquad (1)$$

where

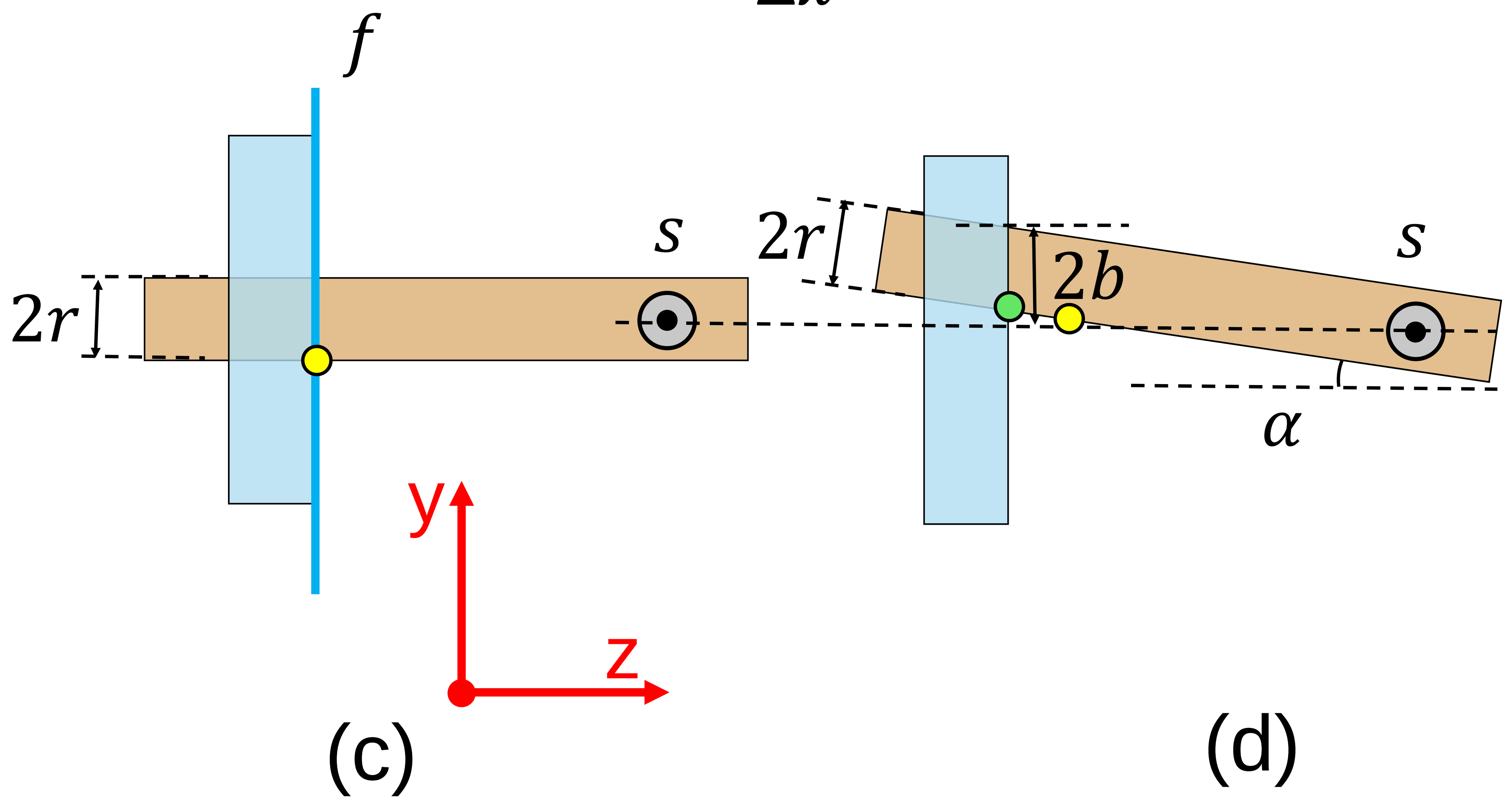
$$k = -tan\varphi \qquad c = -\frac{r}{\cos\varphi} + \Delta x \tan\varphi$$

Denote oval C's equation in the green coordinate system as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

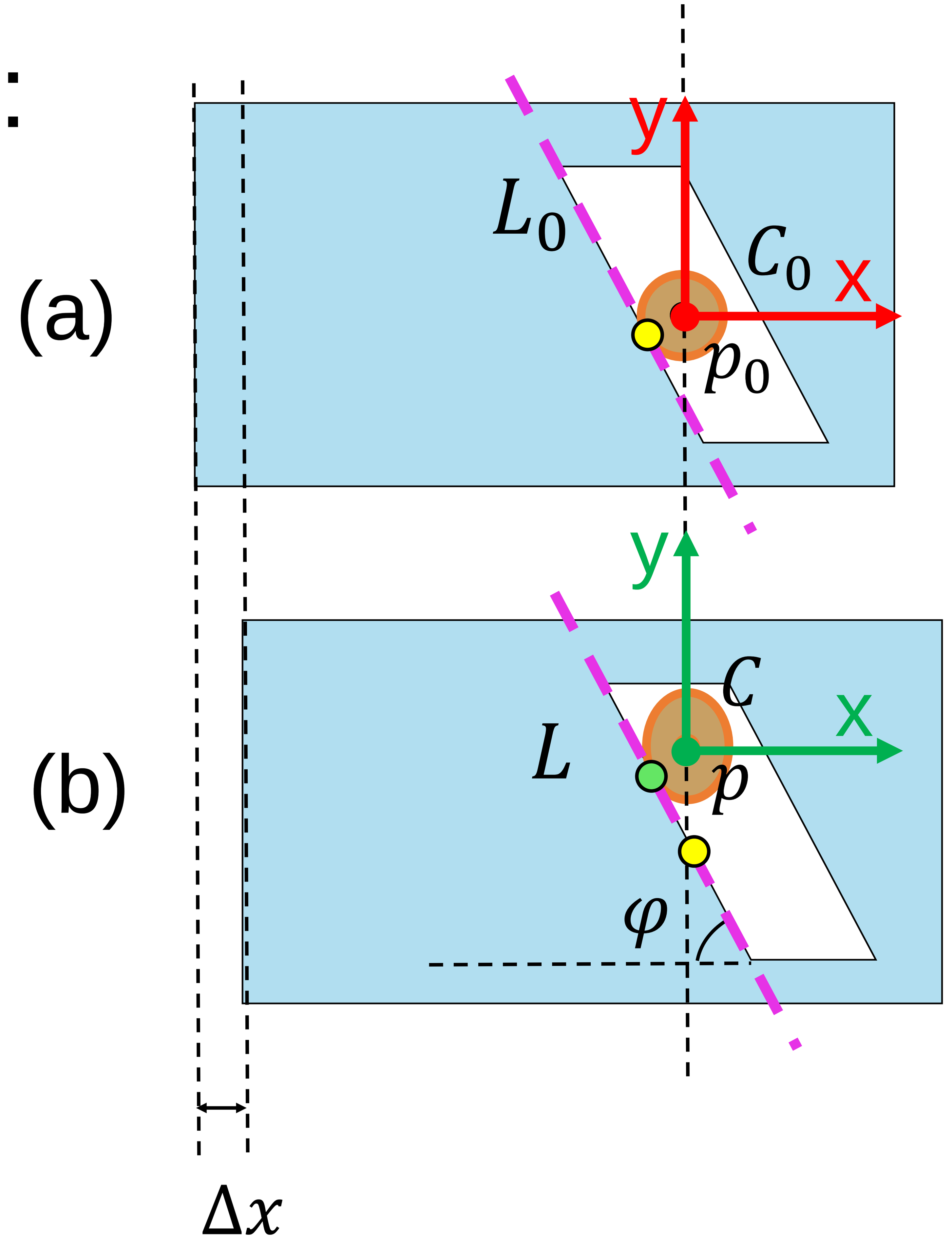
where

$$a = r \qquad b = \frac{r}{\cos\alpha}$$



Denote line L's equation in the green coordinate system as:

$$y = kx + c'$$

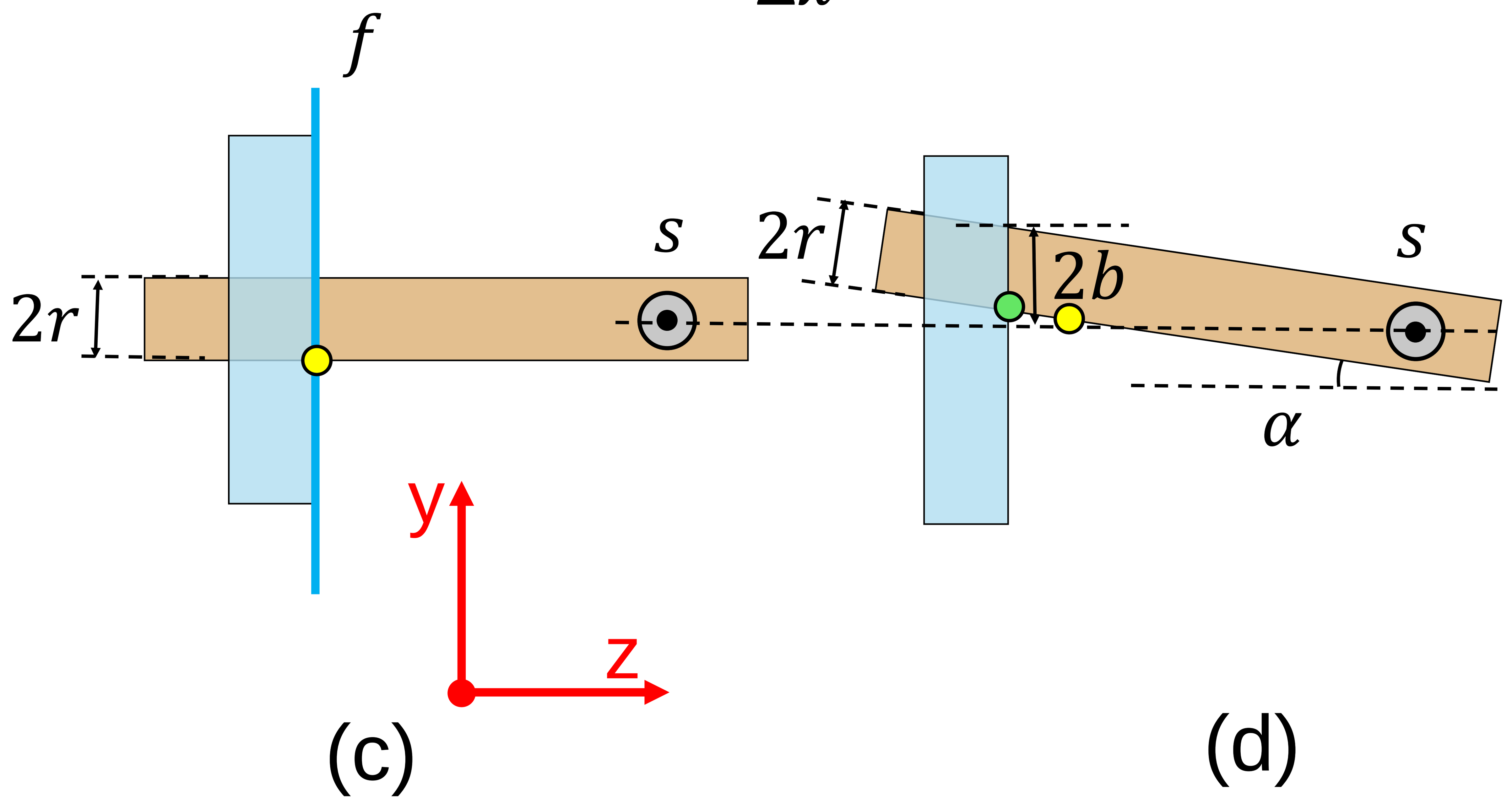


Consider the two equations together:

$$\begin{cases} y = kx + c' \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases}$$

Since L and C are always tangent to each other, the equation set above can have a unique solution. Hence, we have equation (2)

$$a^2k^2 - c'^2 + b^2 = 0 \qquad (2)$$



In addition, the position of p is

$$p_x = p_{0_x}$$
$$p_y = p_{0_y} + (p_{0_z} - s_z)\tan\alpha \qquad (3)$$
$$p_z = p_{0_z}$$

Denote $L = a^2k^2$ $M = k\left(p_{0_x} - s_x\right) + c + s_y - p_{0_y}$ $N = p_{0_z} - s_z$

So, from (1) and (3), we have $c' = M - N \tan \alpha$

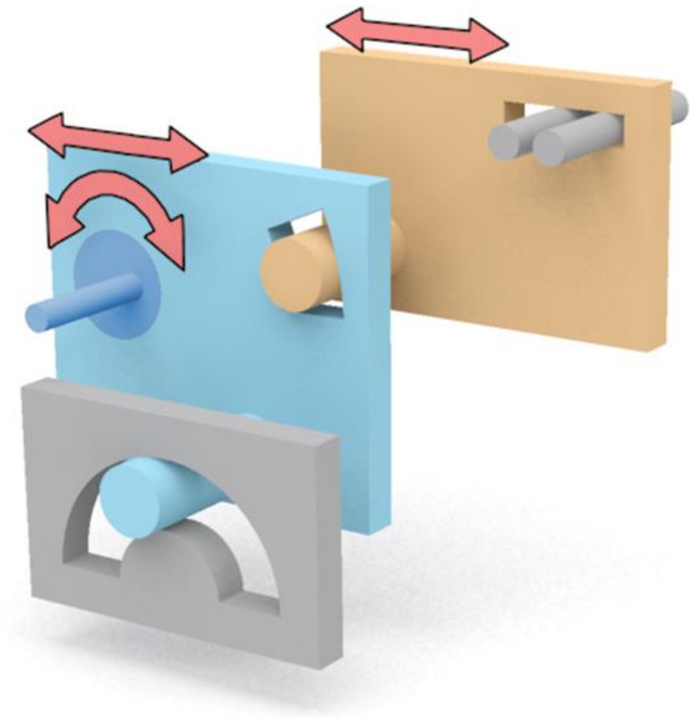
Substituting into (2), we have: $L - (M - N \tan \alpha)^2 + \dfrac{r^2}{cos\, \alpha^2} = 0$

Denote $\theta' = -\tan^{-1} \dfrac{M^2 - L - N^2}{2MN}$

We have: $r^2 - \dfrac{M^2 - L - N^2}{2} = sin(2\alpha + \theta') \sqrt{(\dfrac{M^2 - L - N^2}{2})^2 + (MN)^2}$

Hence,

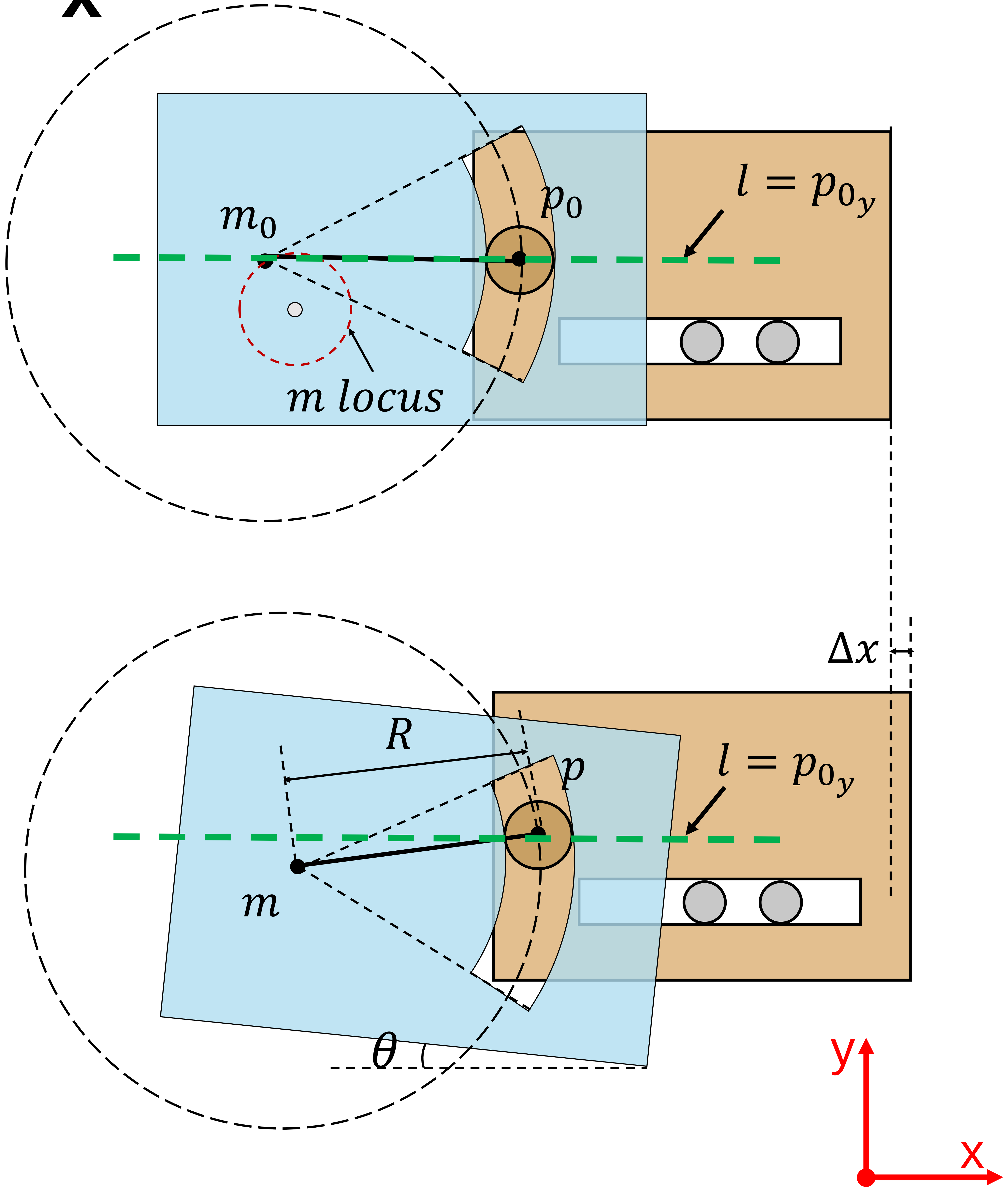
$$\alpha = \frac{sin^{-1}\left(\dfrac{r^2 - \dfrac{M^2 - L - N^2}{2}}{\sqrt{\left(\dfrac{M^2 - L - N^2}{2}\right)^2 + (MN)^2}}\right) - \theta'}{2}$$



#10 $O_zT \rightarrow T_x$

Denote P_d's rotation angle as θ, translation distance along x-axis as Δm_x, and translation distance along y-axis as Δm_y. Denote P_f 's translation distance along x-axis as Δx.

The equation to compute Δx is based on computing the driver-follower joint center p, which is at the intersection between the green line (that is passing through p and aligned with x-axis) and the circle shown on the right figures.



The line's equation is: $y = p_{0_y}$

The circle is centered at m and has a radius

$$R = \|p_0 - m_0\|$$

Putting the two equations together, we have

$$\begin{cases} p_y = p_{0_y} \\ (p_x - m_x)^2 + (p_y - m_y)^2 = R^2 \end{cases}$$

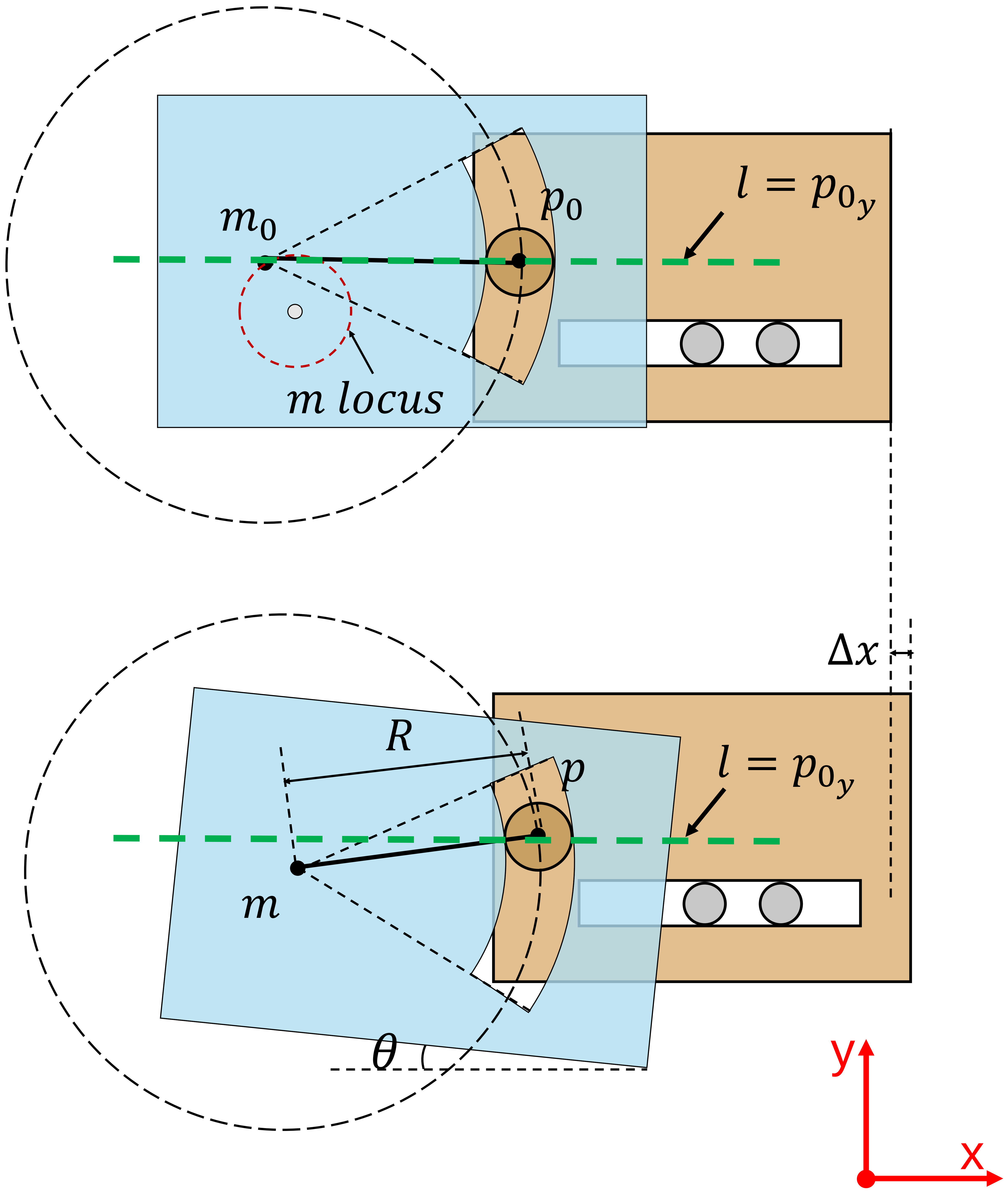
By solving above two equations, we get

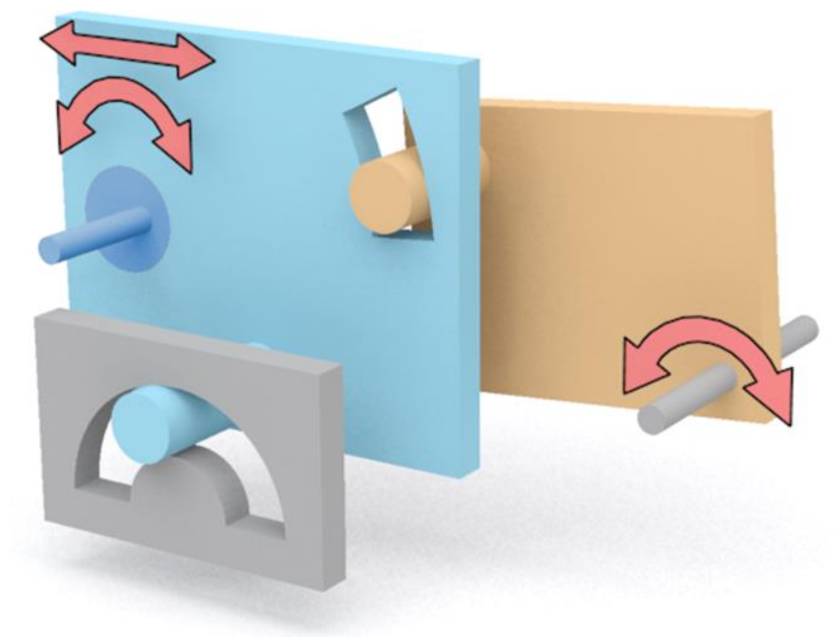
$$\Delta x = p_x - p_{0_x}$$

where

$$p_x = \sqrt{R^2 - (p_{0_y} - m_y)^2}$$

$$m_y = m_{0_y} + \Delta m_y$$

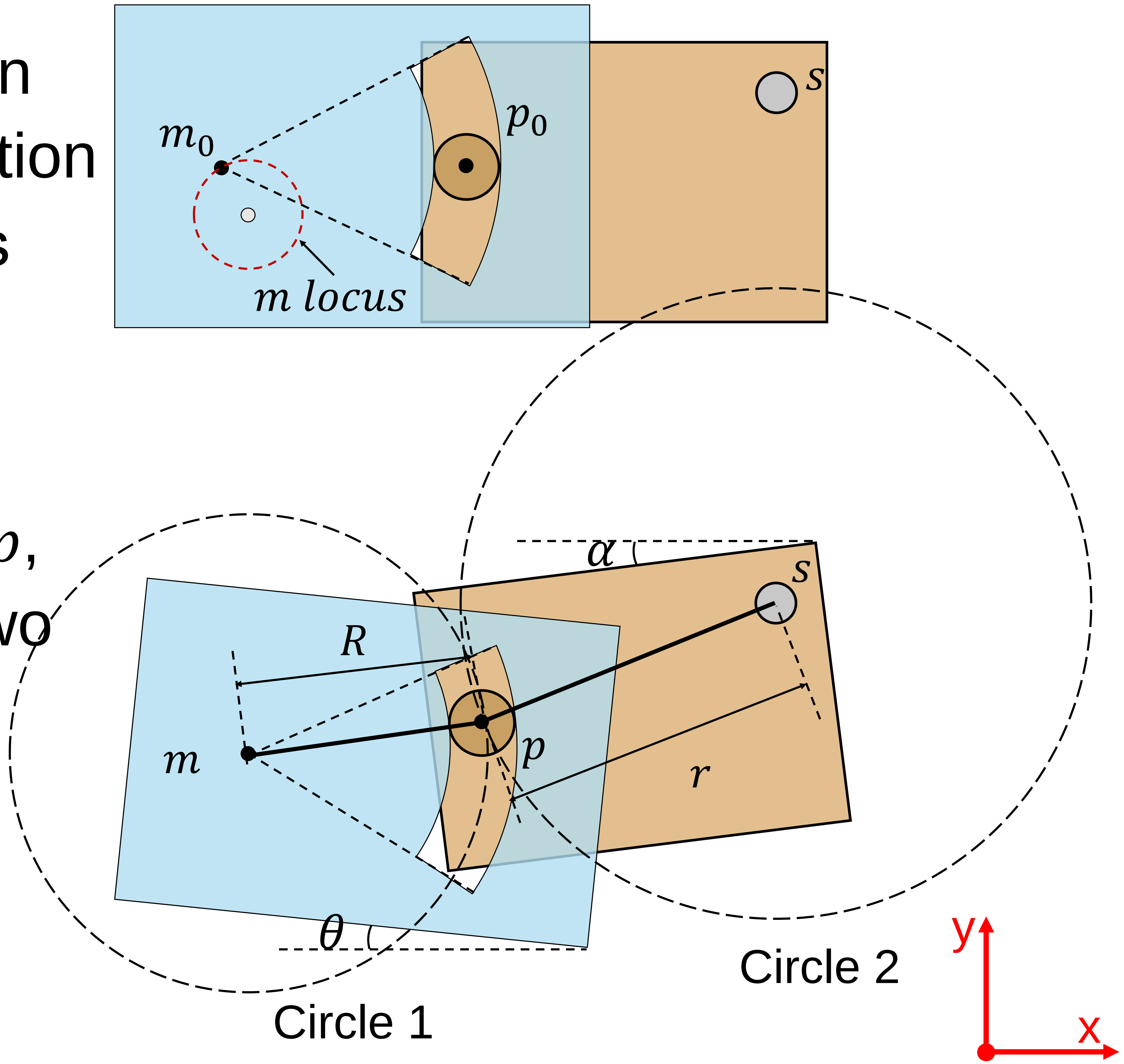




#11 $O_z T \rightarrow O_z$

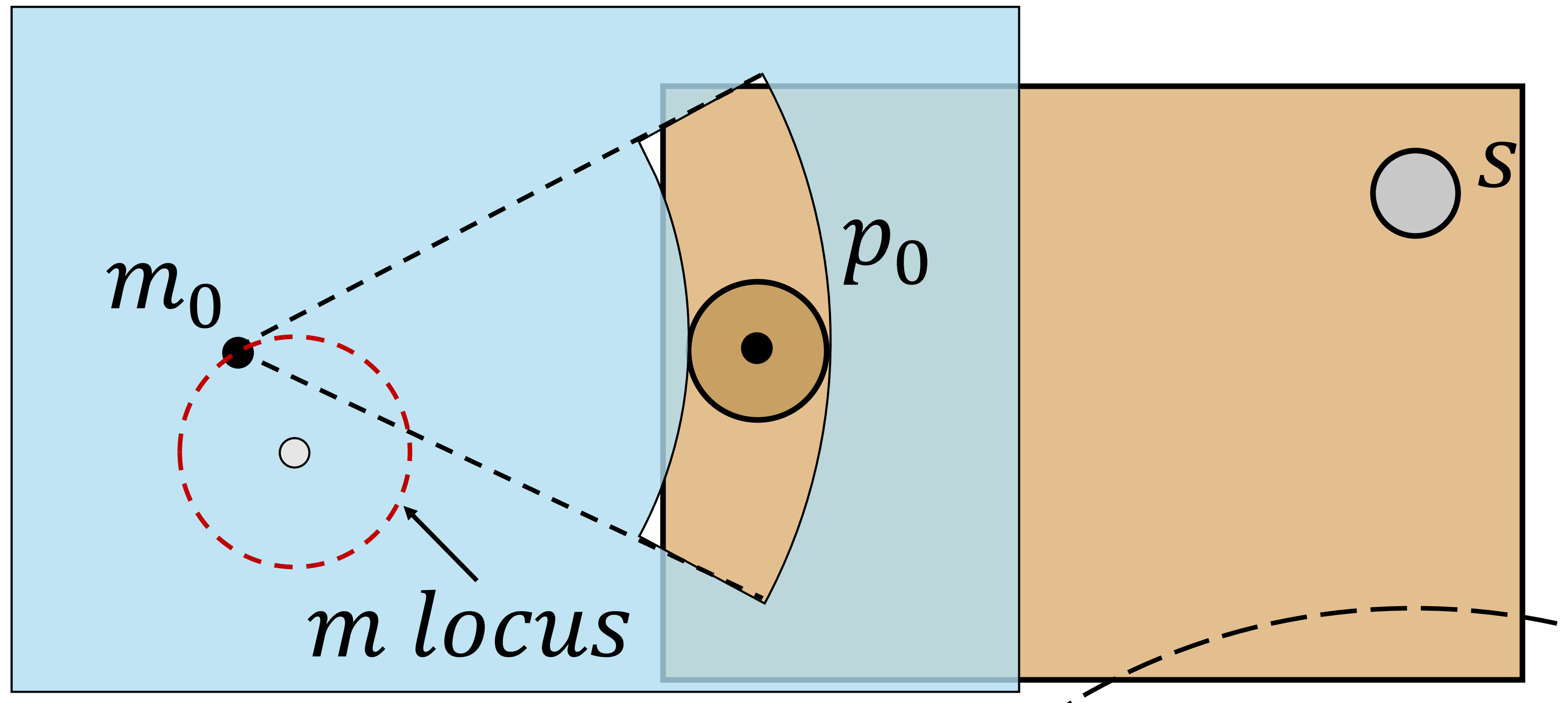
Denote P_d's rotation angle as θ, translation distance along x-axis as Δm_x, and translation distance along y-axis as Δm_y. Denote P_f's rotation angle as α.

The equation to compute α is based on computing the driver-follower joint center p, which is at the intersection between the two circles.



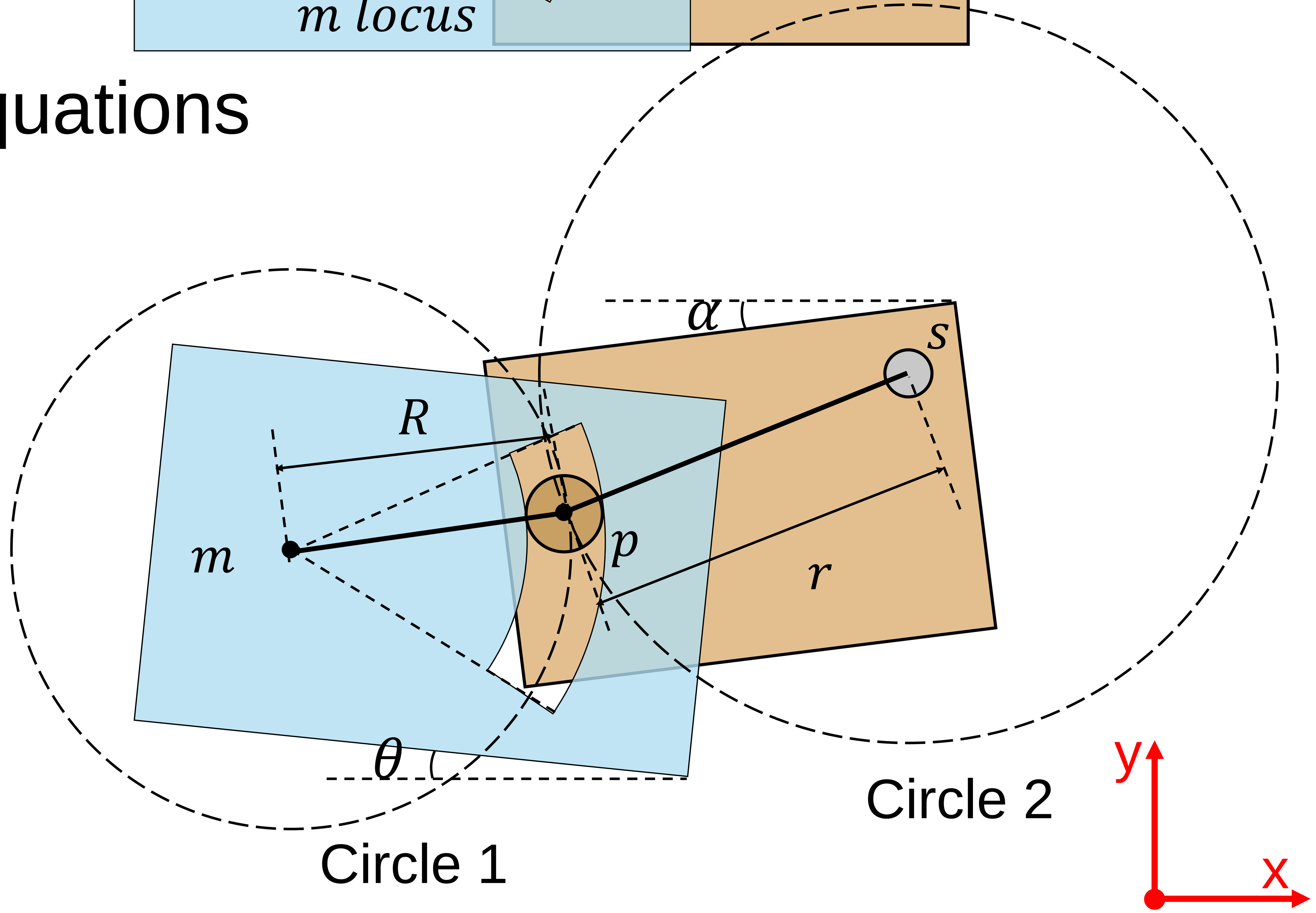
The 2 circles that p locates are

Circle 1 : center m ; radius $R = ||p_0 - m_0||$
Circle 2: center s ; radius $r = ||p_0 - s||$



Hence, p must satisfy the following two equations

$$\begin{cases}(p_x - m_x)_2 + (p_y - m_y)^2 = R^2 \\ (p_x - s_x)^2 + (p_y - s_y)^2 = r^2\end{cases}$$



Note that m_0 and m are computed in the same way as in #2 $R_z \rightarrow O_z$

By solving above two equations, we get

$$p_y = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + s_y$$

$$p_x = \frac{r^2 - R^2 + {m_x{'}}^2 + {m_y{'}}^2 - 2m_y{'}(p_y - s_y)}{2m_x{'}} + s_x$$

where

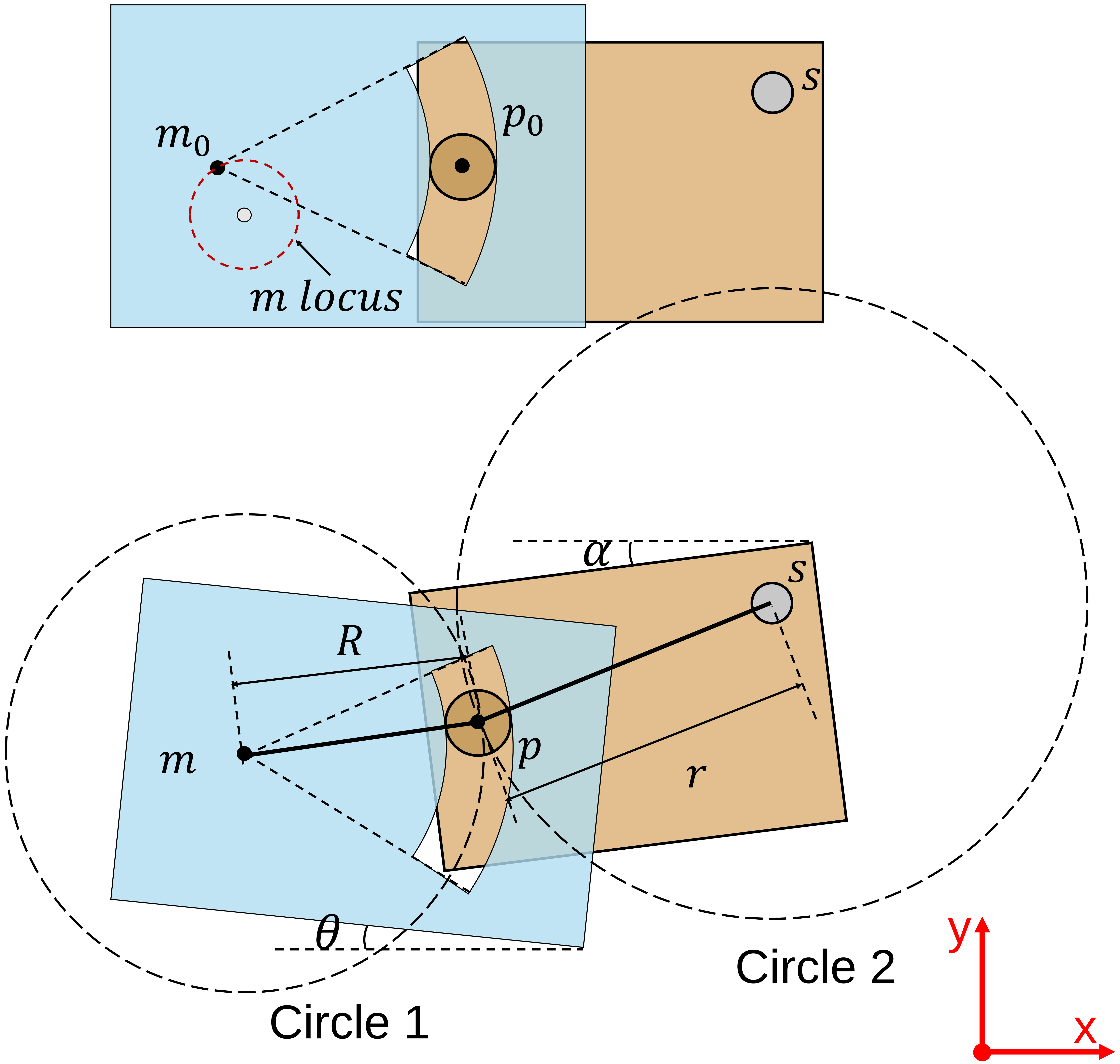
$$m_x{'} = m_x - s_x$$

$$m_y{'} = m_y - s_y$$

$$a = 4({m_x{'}}^2 + {m_y{'}}^2)$$

$$b = -4m_y{'}(r^2 - R^2 + {m_x{'}}^2 + {m_y{'}}^2)$$

$$c = (r^2 - R^2 + {m_x{'}}^2 + {m_y{'}}^2)^2 - (2m_x{'}R)^2$$



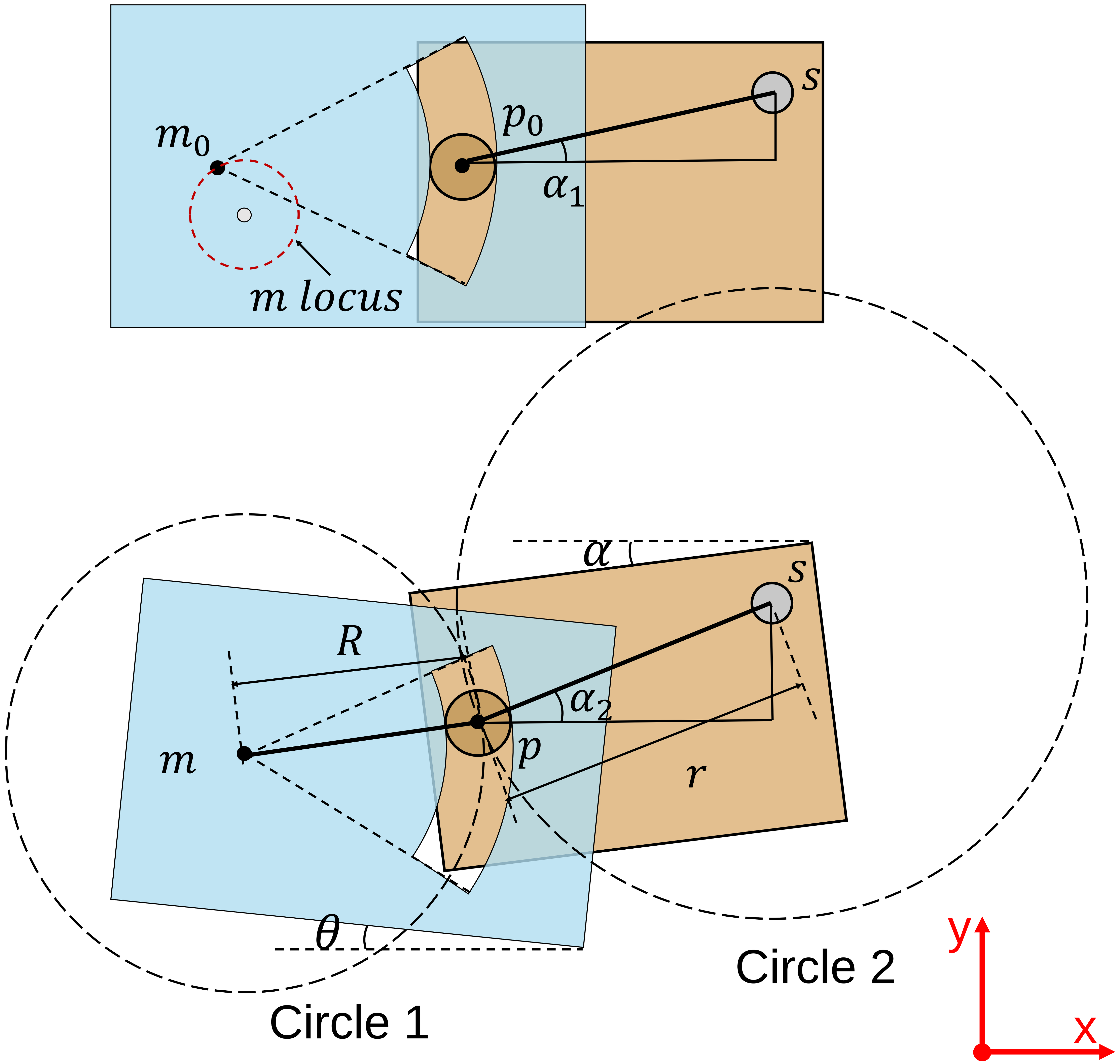
Based on the calculated p, we have

$$\alpha = \alpha_2 - \alpha_1$$

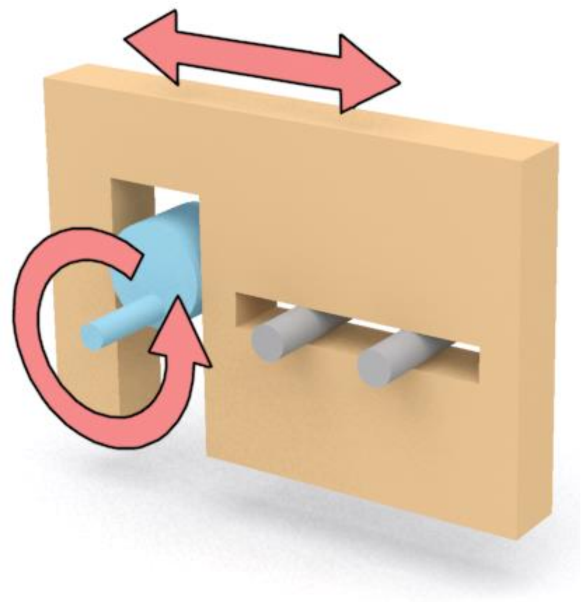
where

$$\alpha_1 = \tan^{-1}\left(\frac{s_y - p_{0_y}}{s_x - p_{0_x}}\right)$$

$$\alpha_2 = \tan^{-1}\left(\frac{s_y - p_y}{s_x - p_x}\right)$$



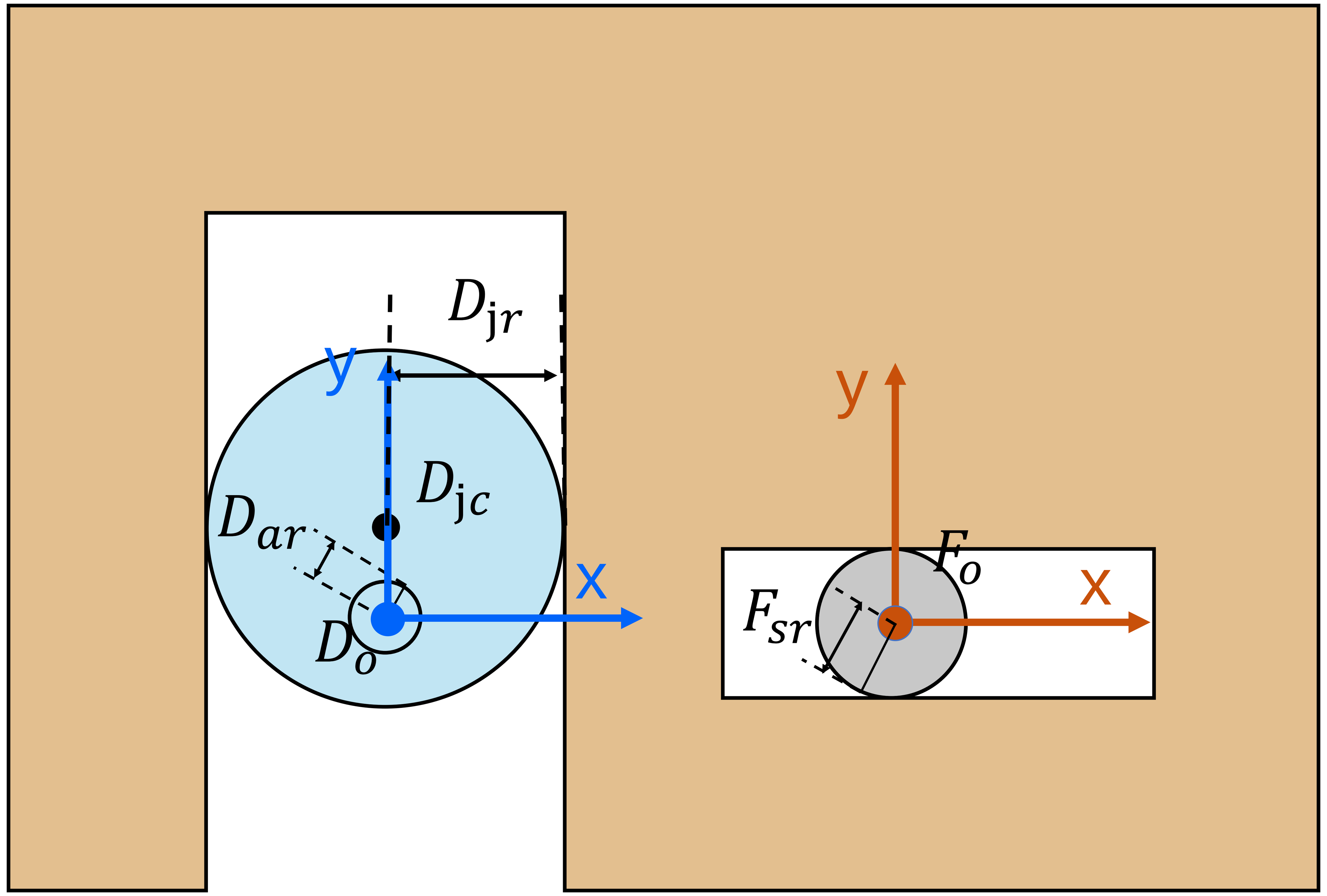
Part 2:
Default Geometric Parameters of Elemental Mechanisms



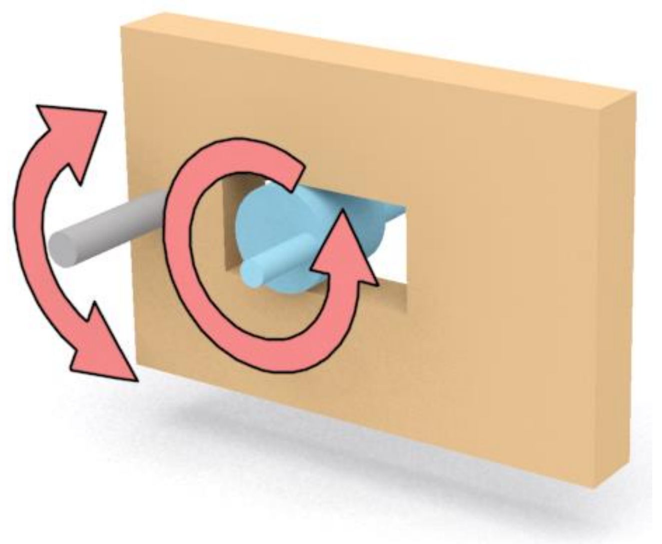
#1 $R_z \rightarrow T_x$

- D_o: (0.00, 0.00) // driver origin (rotation center)
- D_{ar}: 0.03 // driver rotation axis radius
- D_{jc}: (0.00, 0.06) // driver (cam) geometric center
- D_{jr}: 0.12 // driver (cam) radius

- F_o: (0.30, 0.00) // follower support center
- F_{sr}: 0.035 // follower support axis radius



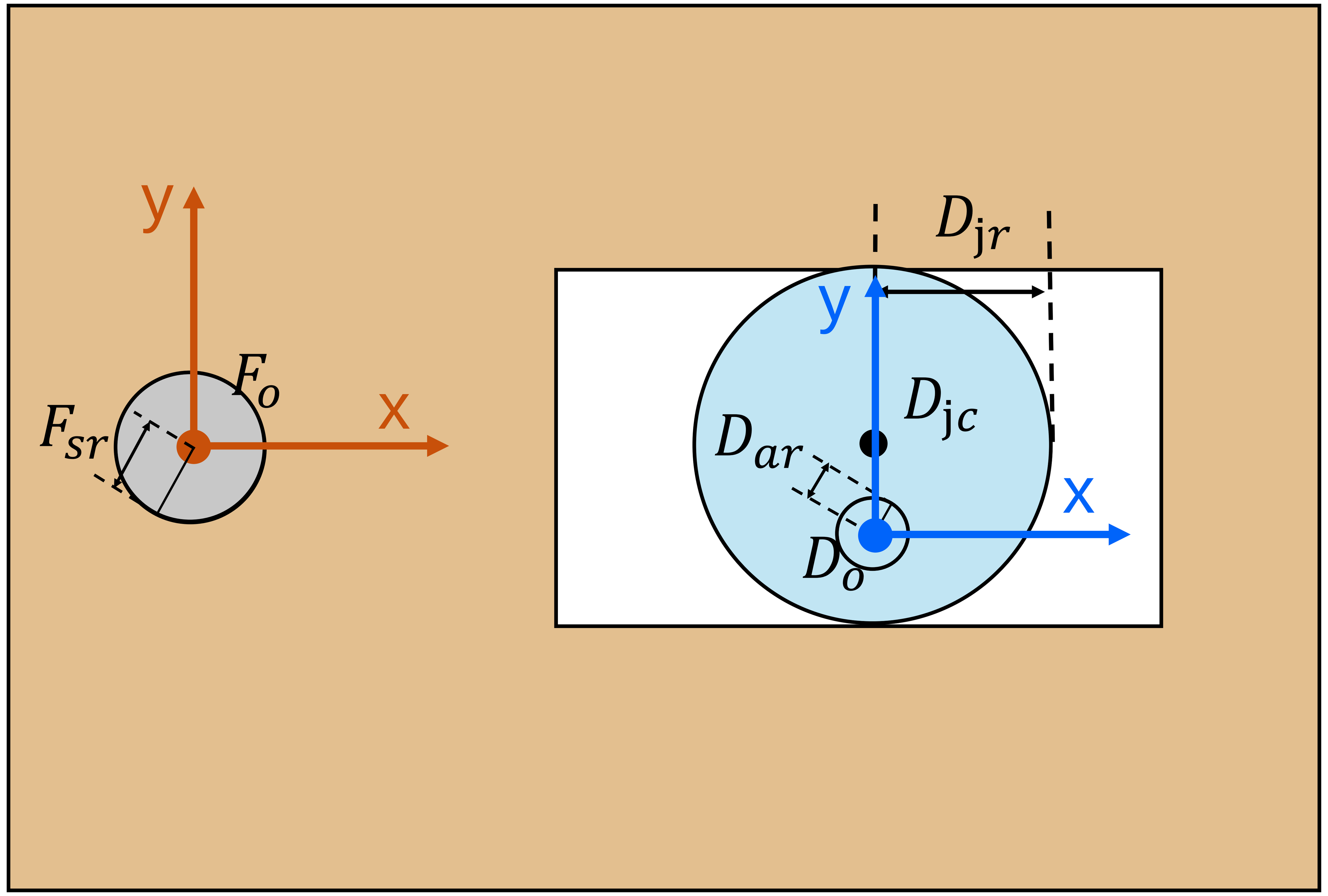
Note:
1) Coordinate of F_0 is relative to the driver's local coordinate (blue one);
2) Coordinates of follower's all other positional parameters are relative to the follower's local coordinate (orange one);
Above two rules apply to all elemental mechanisms.

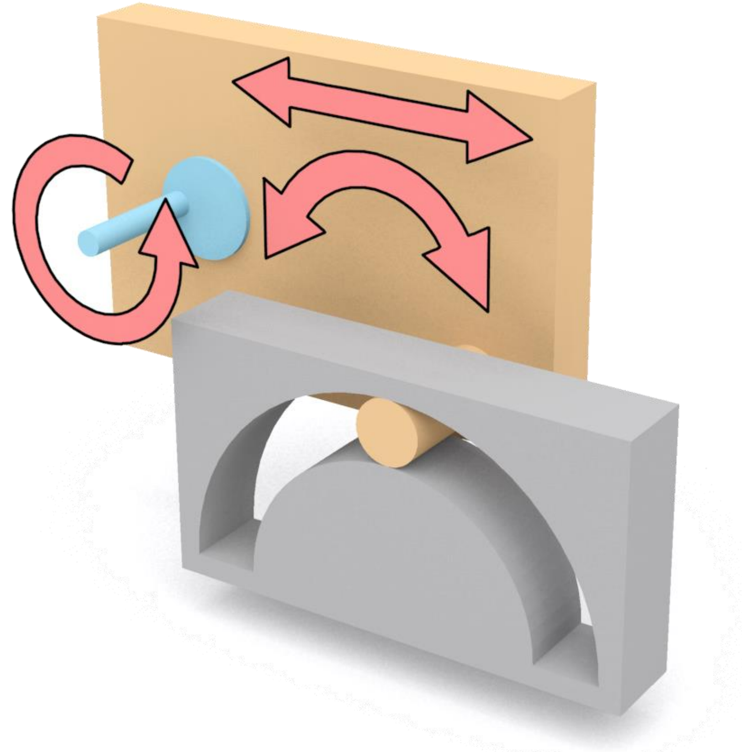


#2 $R_z \rightarrow O_z$

- D_o: (0.00, 0.00)
- D_{ar}: 0.03
- D_{jc}: (0.00, 0.06)
- D_{jr}: 0.12

- F_o: (-0.40, 0.10)
- F_{sr}: 0.03

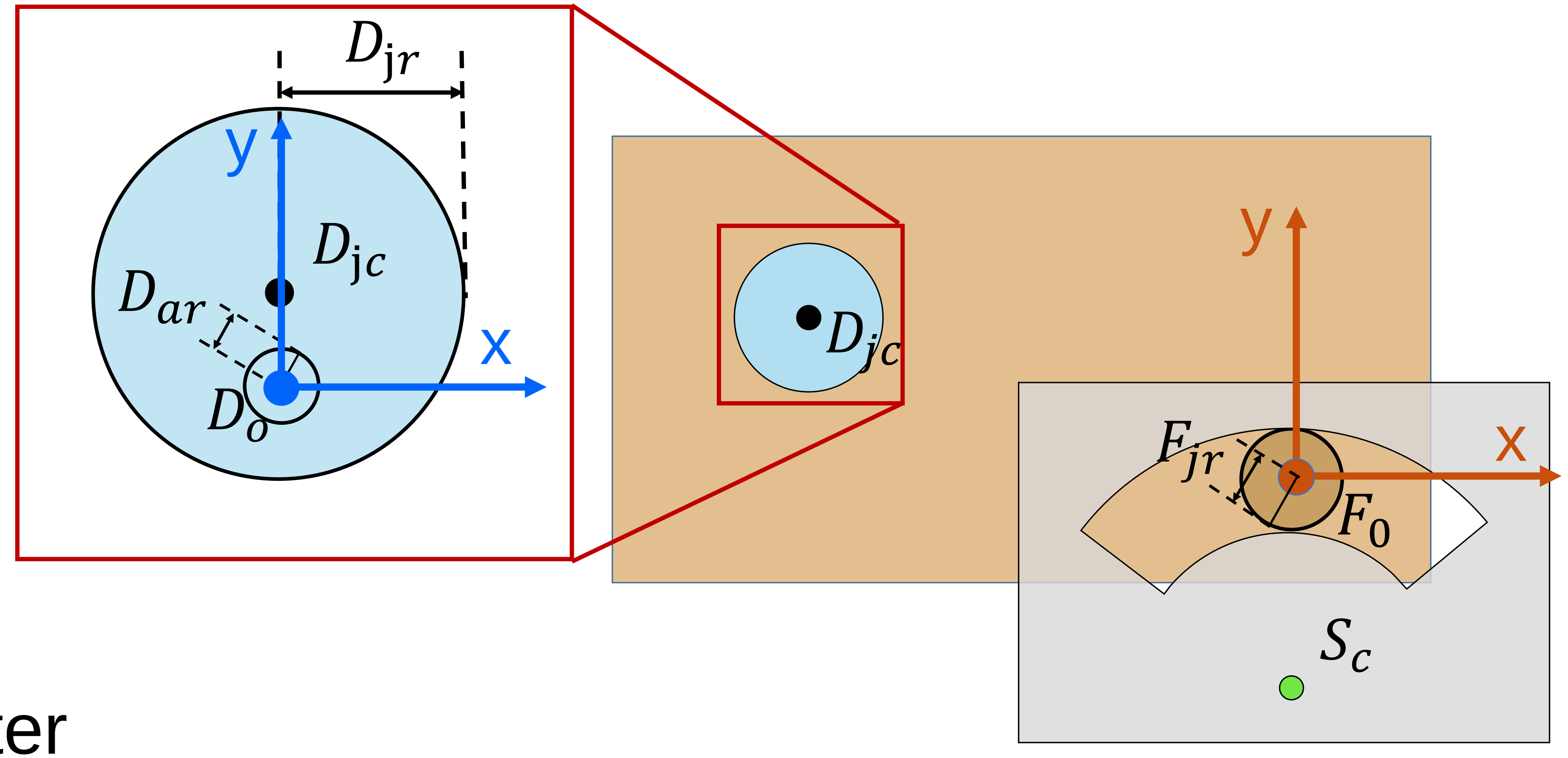


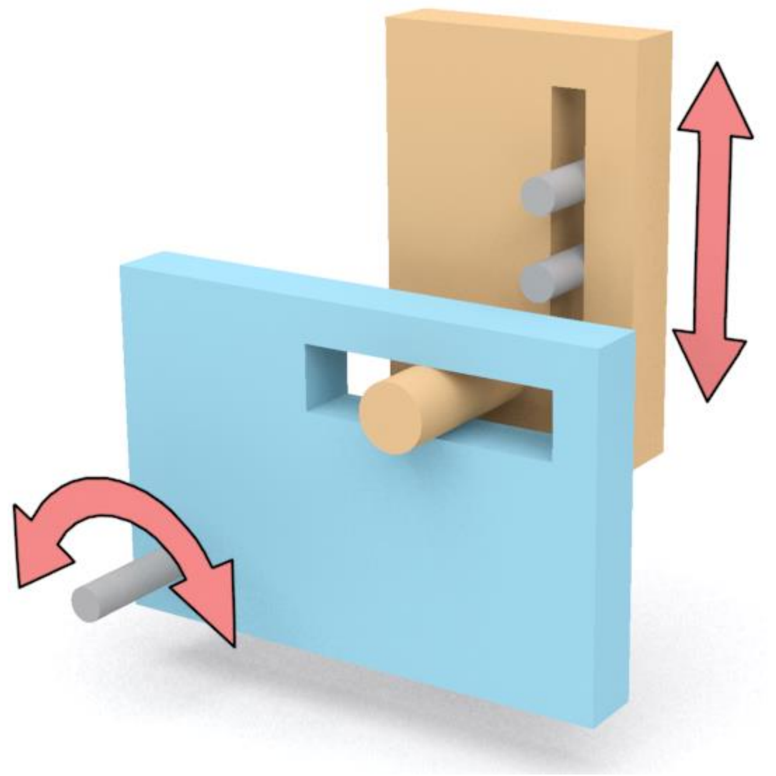


#3 $R_z \rightarrow O_z T$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03
- D_{jc}: (0.00, 0.04)
- D_{jr}: 0.12

- F_0: (0.60, -0.30)
- F_{jr}: 0.075
- S_c: (-0.10, -0.20, 0.10) // support center

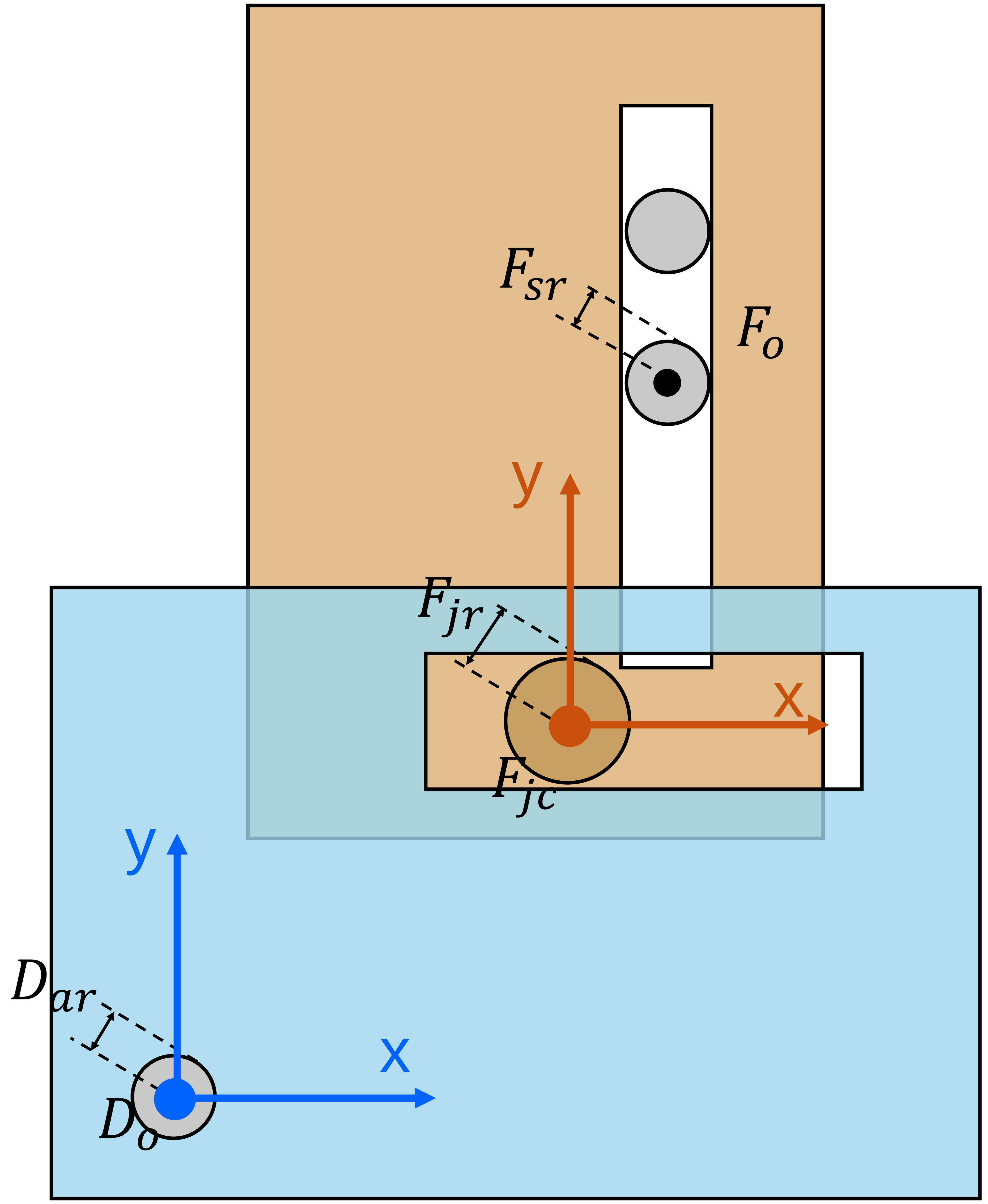


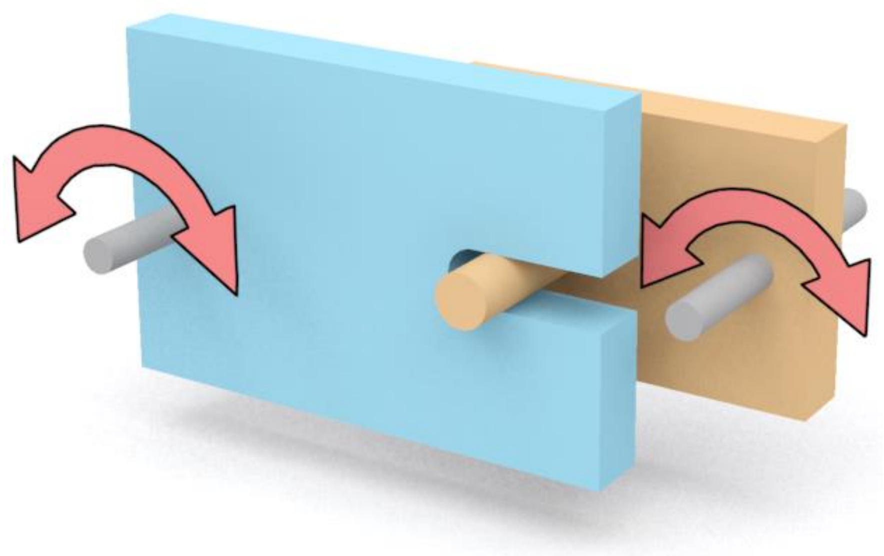


#4 $O_z \rightarrow T_y$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03

- F_o: (0.80, 0.60, -0.40)
- F_{jc}: (-0.2, -0.3)
- F_{jr}: 0.07
- F_{sr}: 0.035

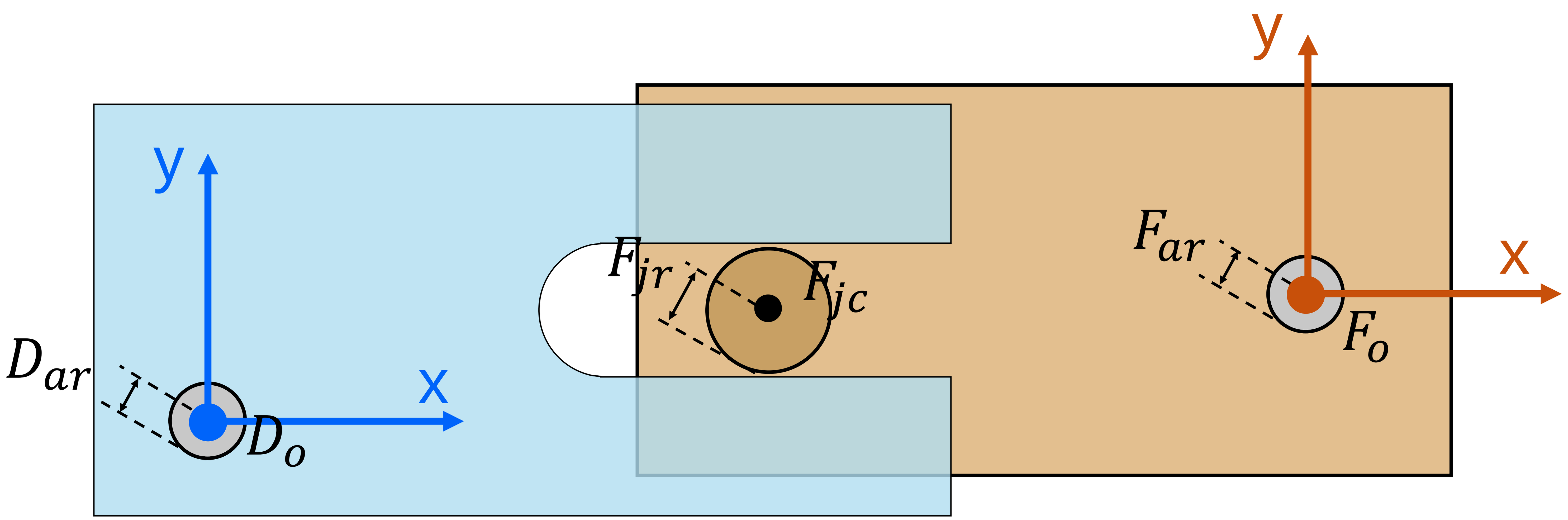


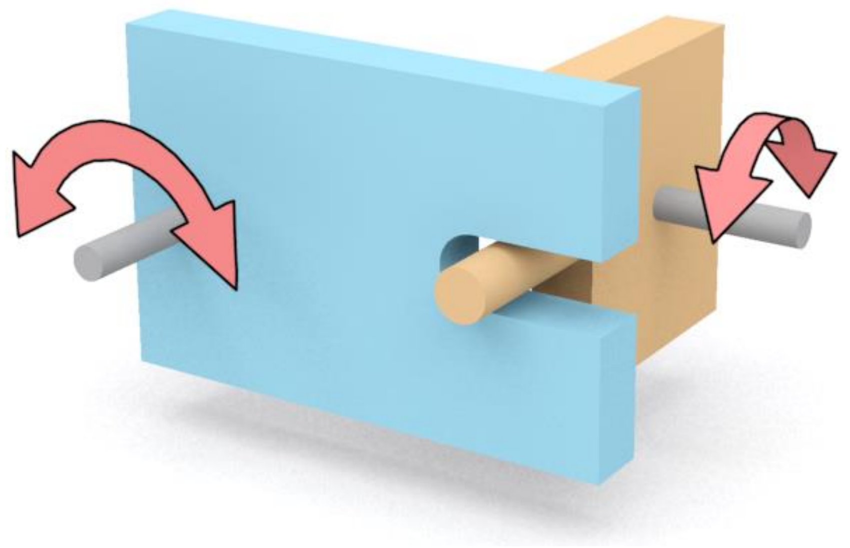


#5 $O_z \rightarrow O_z$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03

- F_o: (1.30, 0.20, -0.30)
- F_{ar}: 0.03
- F_{jc}: (-0.60, 0.00)
- F_{jr}: 0.06

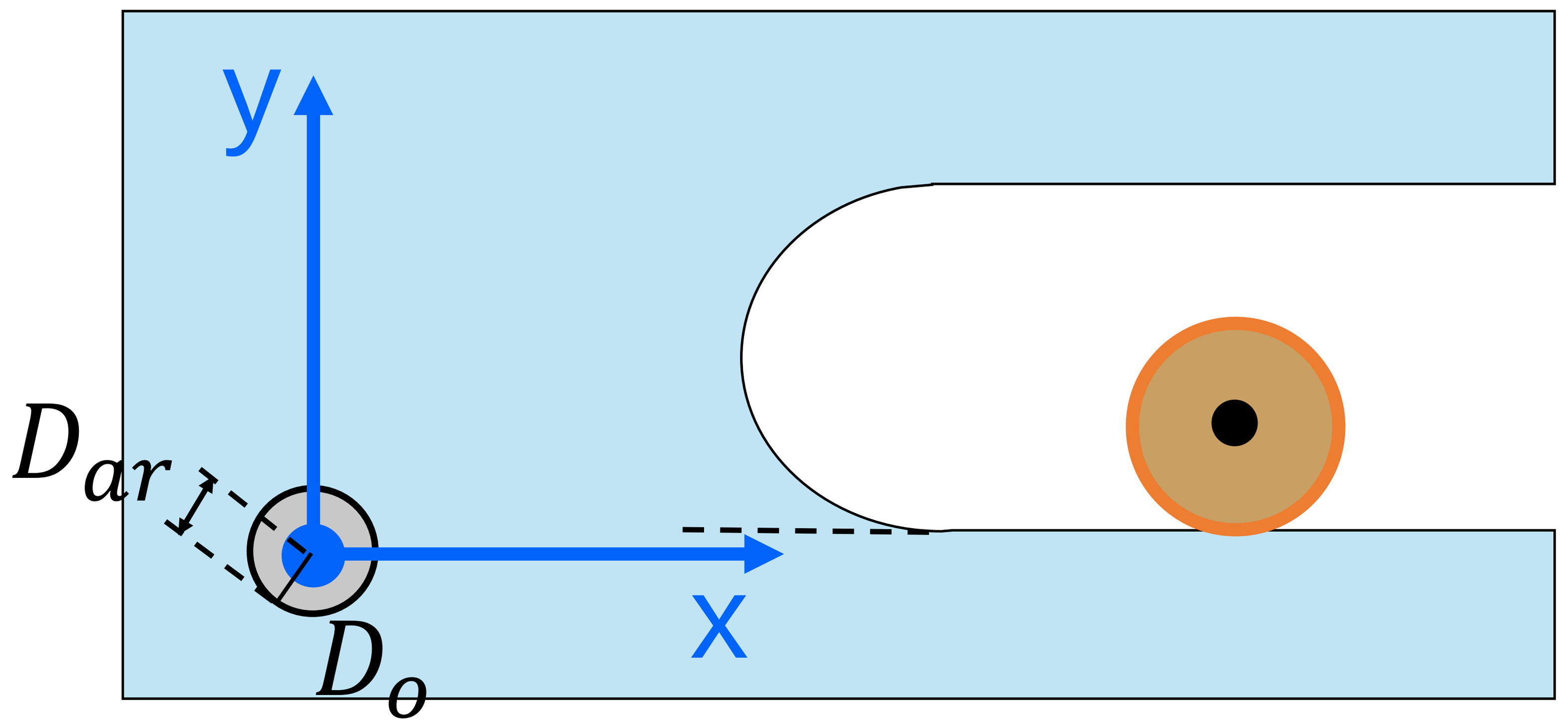


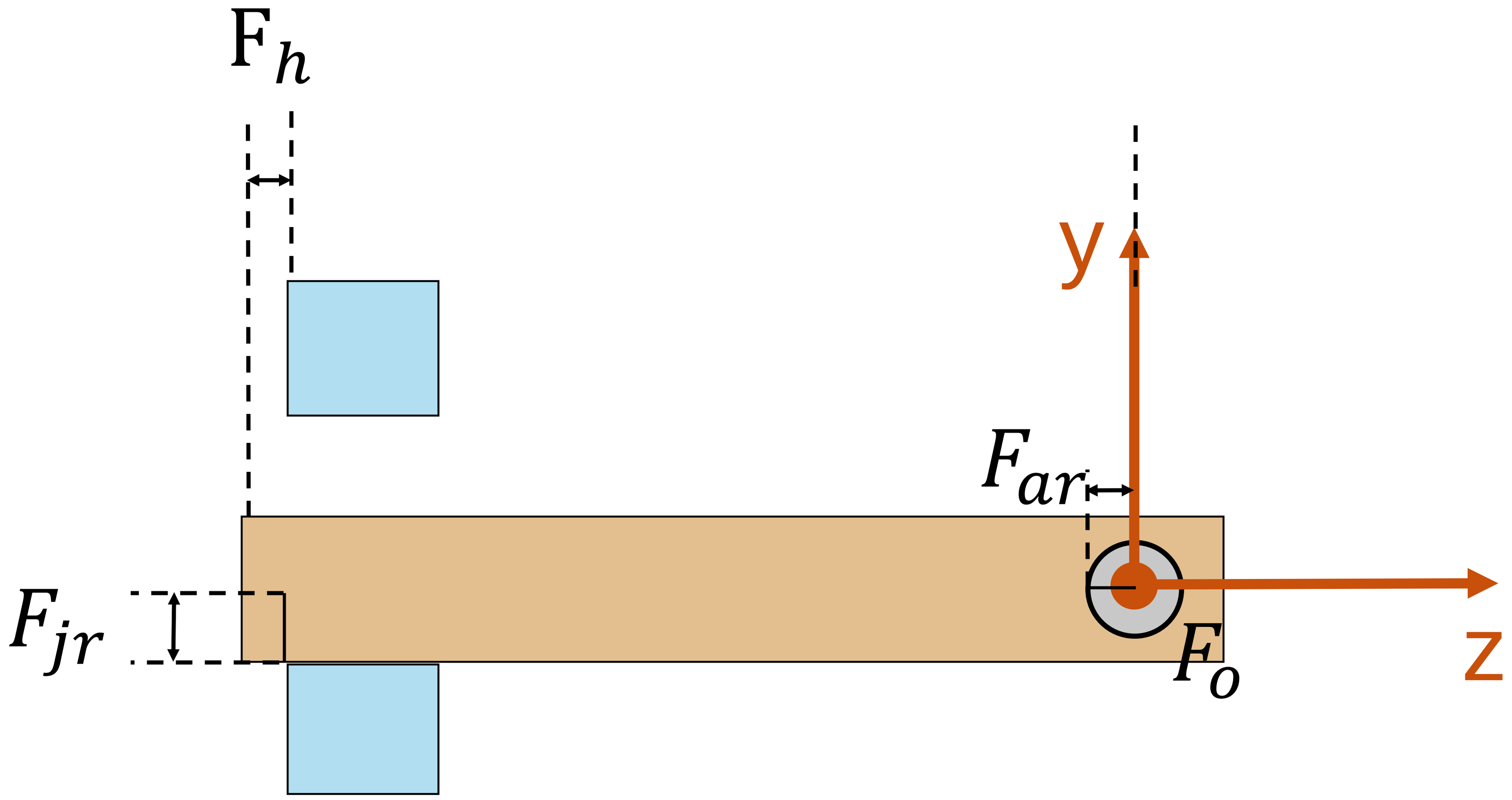


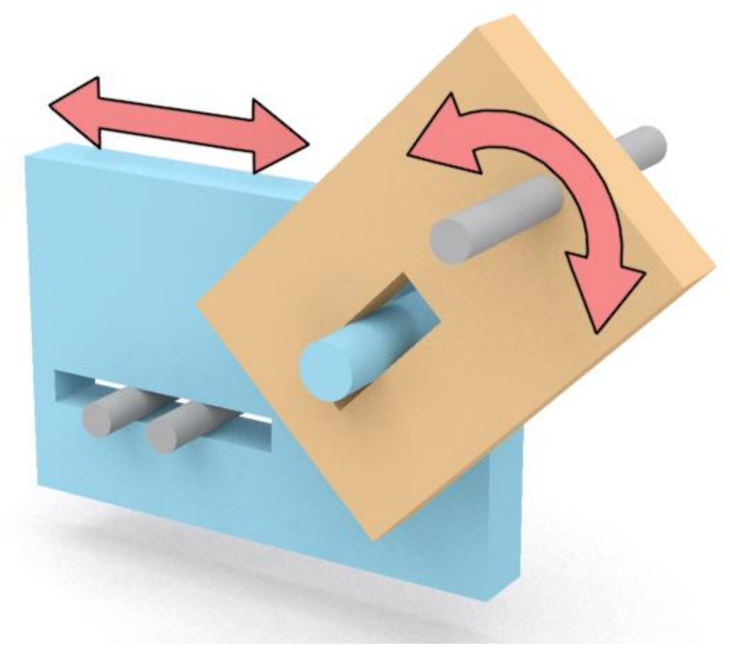
#6 $O_z \rightarrow O_x$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03

- F_o: (0.70, 0.00, -0.80)
- F_{ar}: 0.03
- F_{jr}: 0.06
- F_h: 0.05



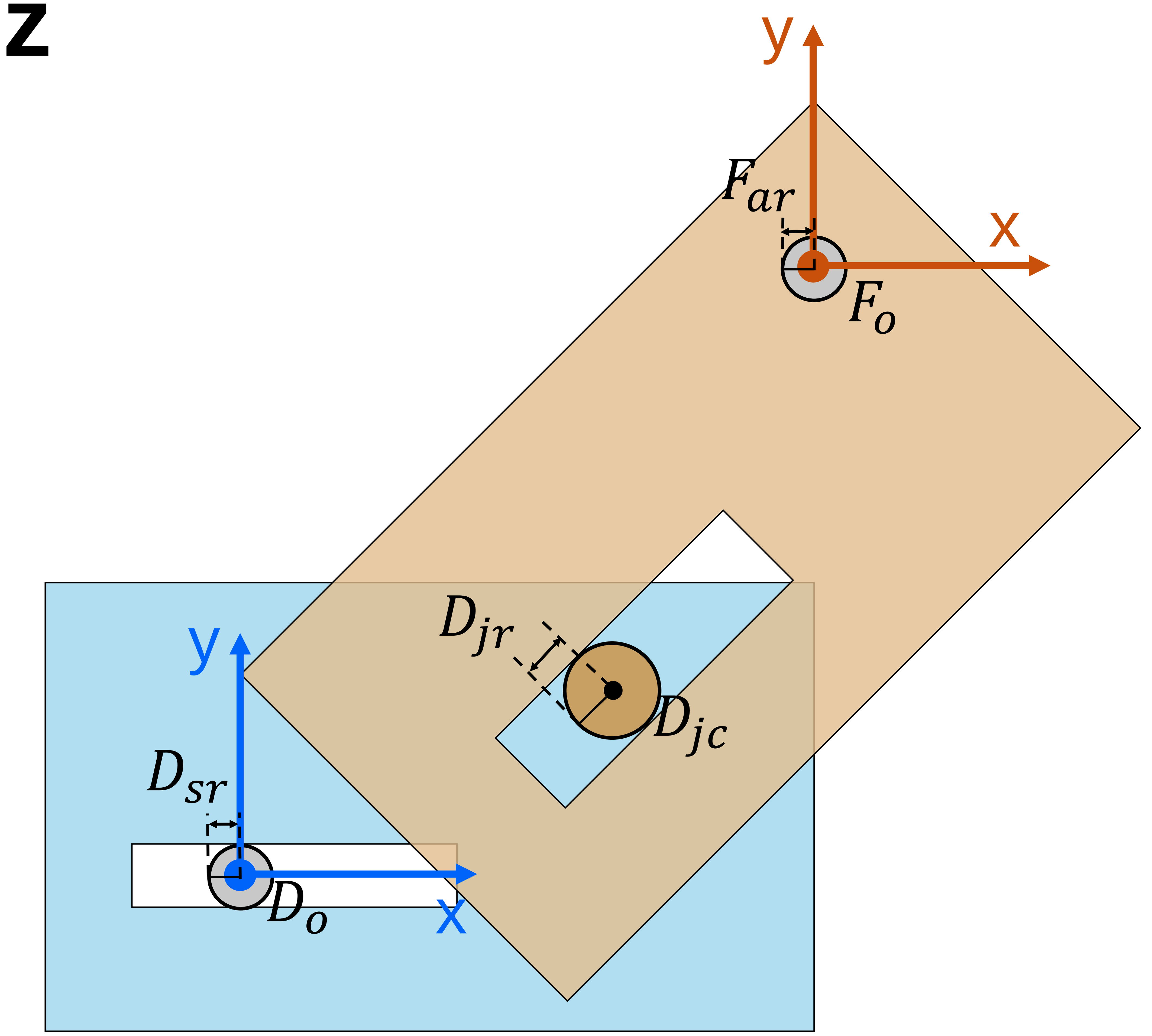


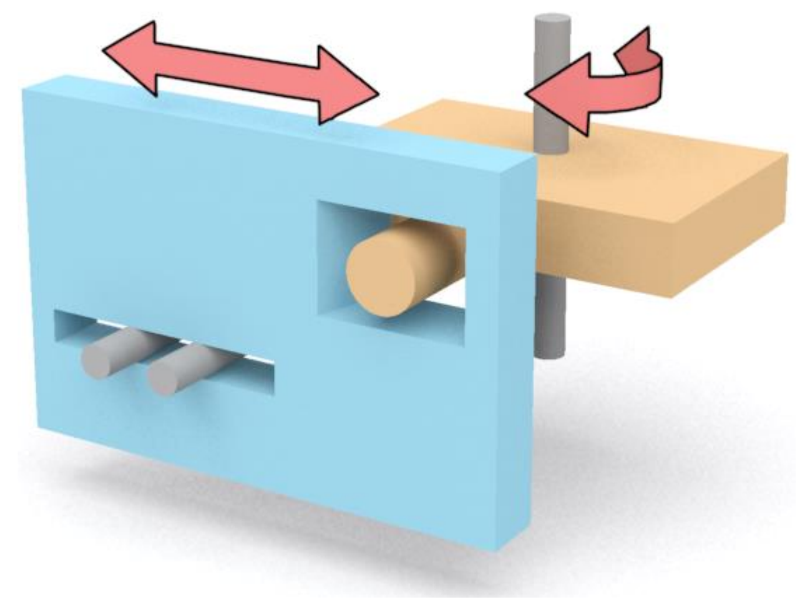


#7 $T_x \rightarrow O_z$

- D_o: (0.00, 0.00, 0.00)
- D_{jc}: (0.50, 0.25)
- D_{jr}: 0.04
- D_{sr}: 0.035

- F_o: (0.70, 0.55, 0.30)
- F_{ar}: 0.035

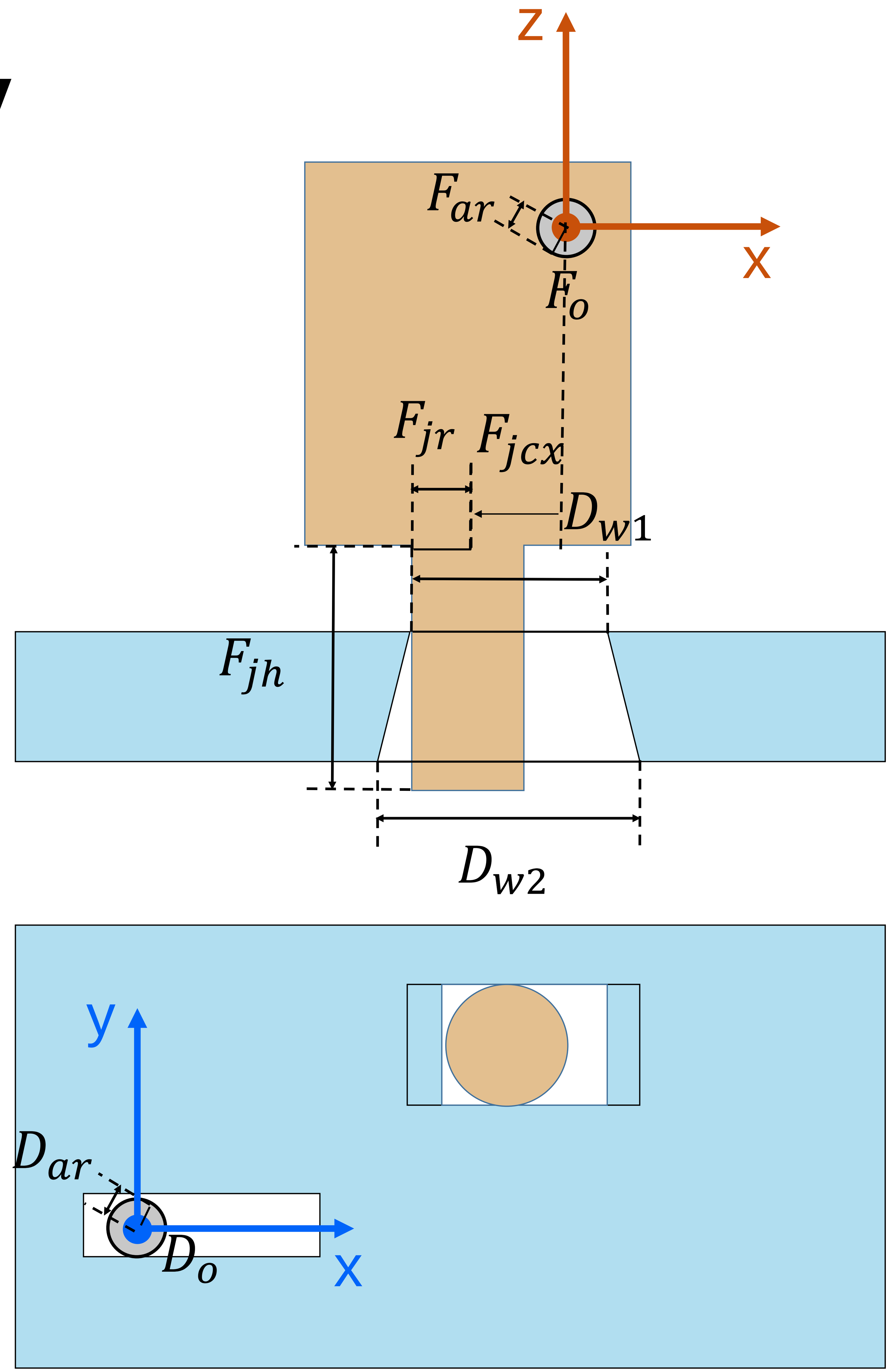


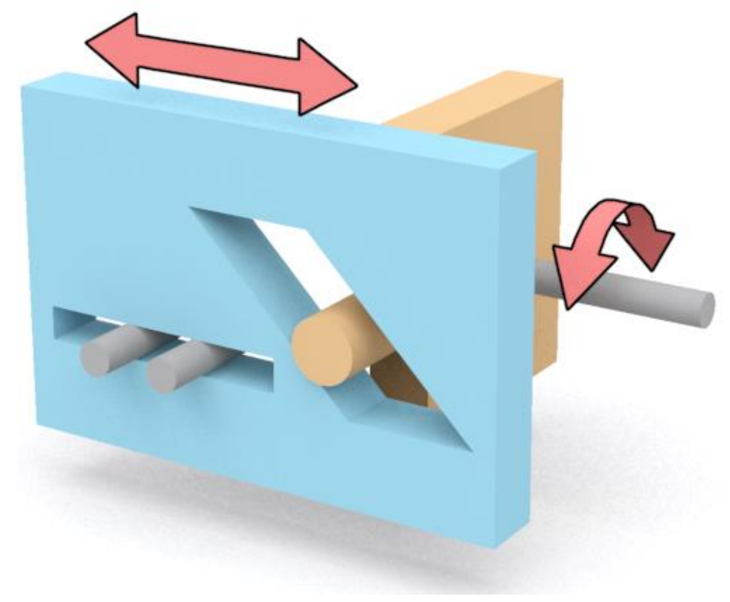


#8 $T_x \rightarrow O_y$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03
- D_{w1}: 0.20
- D_{w2}: 0.30

- F_o: (0.55, 0.30, -0.60)
- F_{ar}: 0.035
- F_{jcx}: 0.05
- F_{jr}: 0.06
- F_{jh}: 0.40

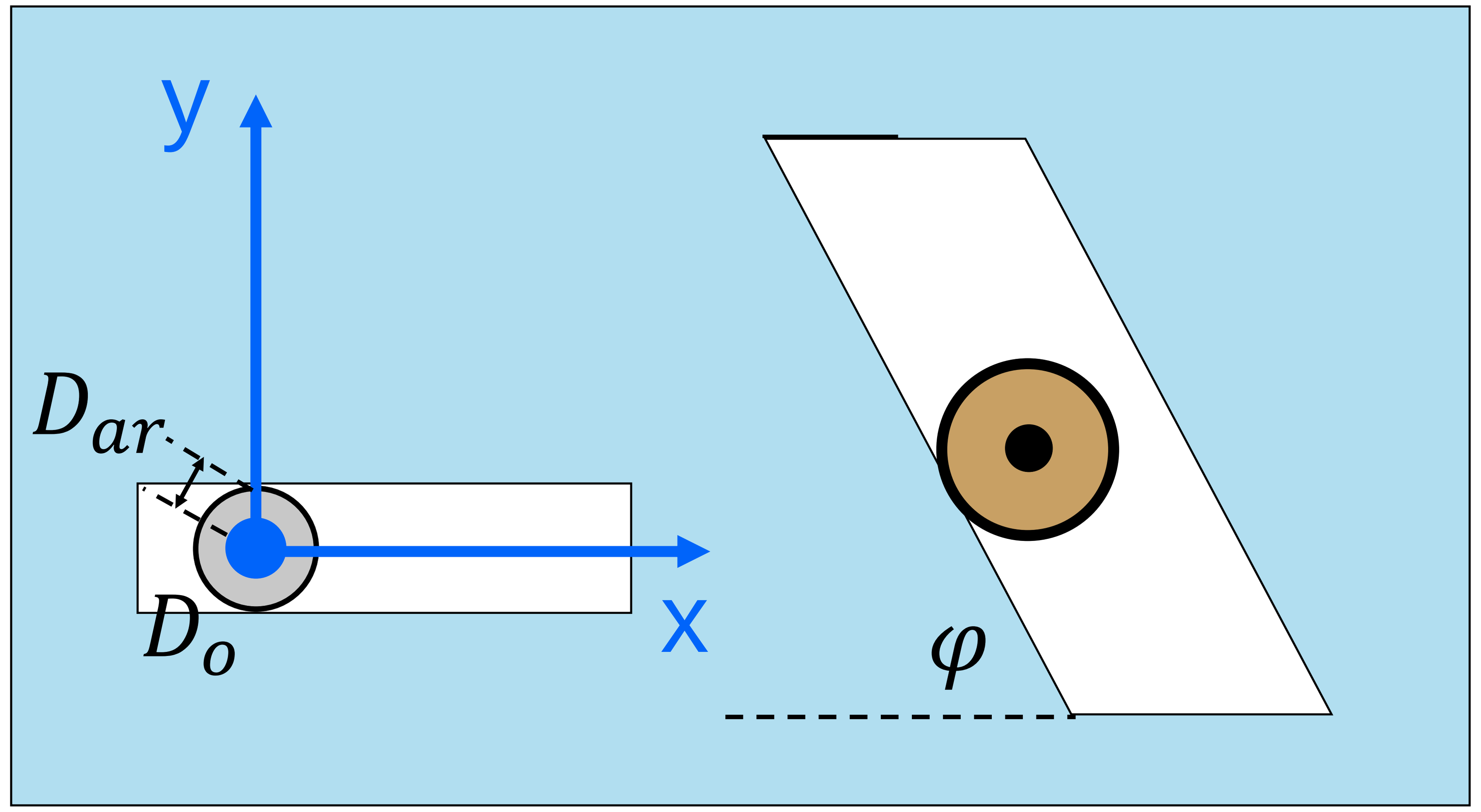


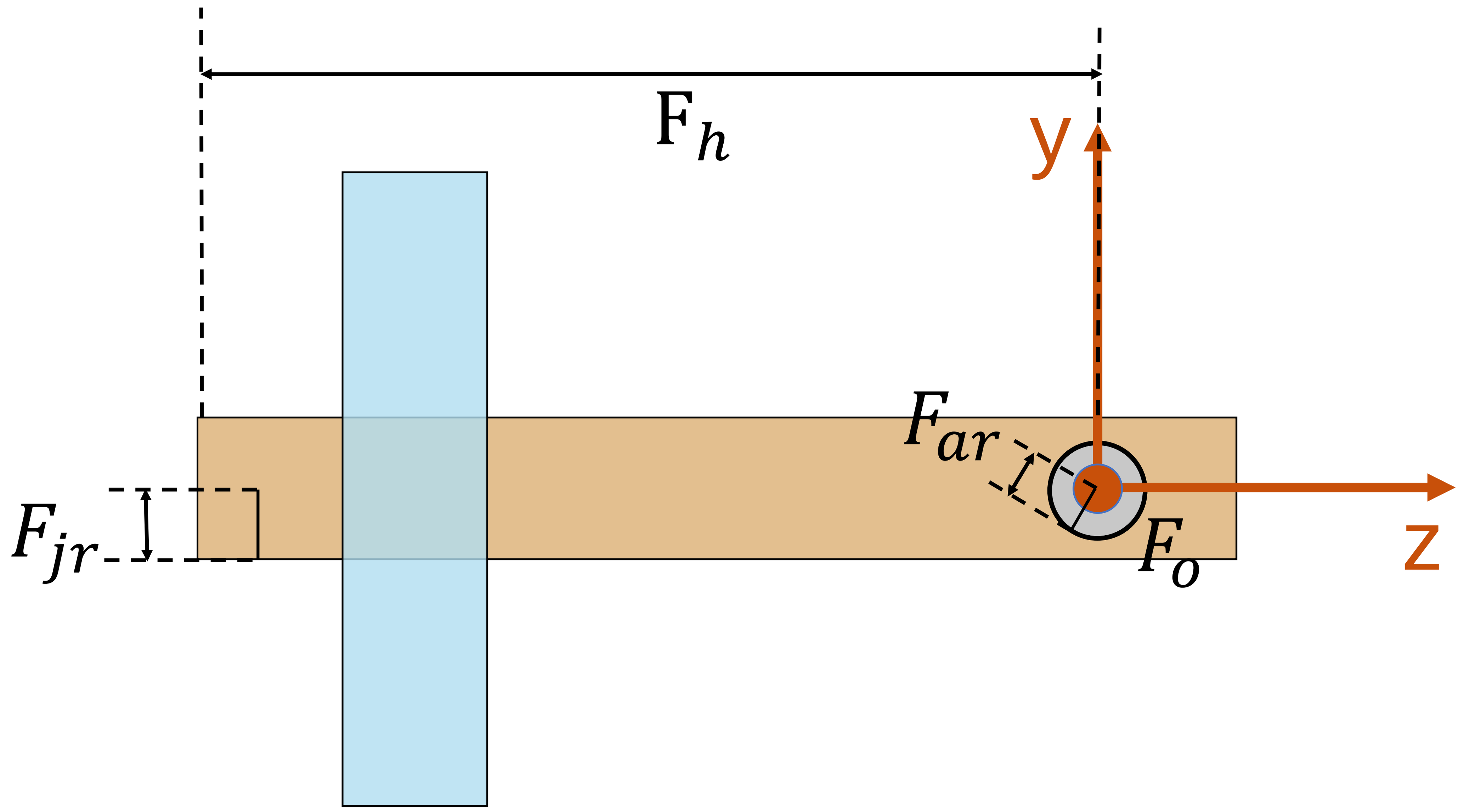


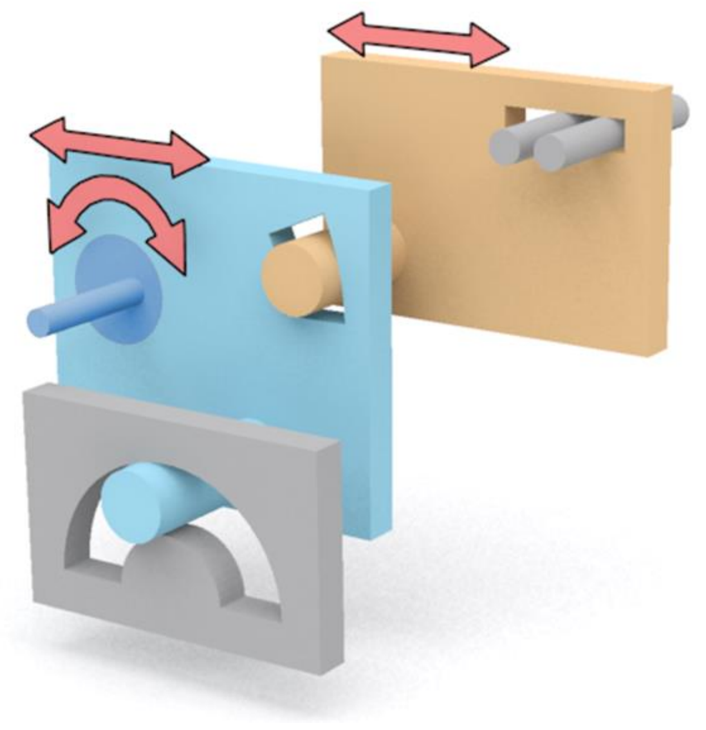
#9 $T_x \rightarrow O_x$

- D_o: (0.00, 0.00, 0.00)
- D_{ar}: 0.03
- φ: $\frac{\pi}{3}$

- F_o: (0.70, 0.55, -0.30)
- F_{ar}: 0.035
- F_{jr}: 0.04
- F_h: 0.40



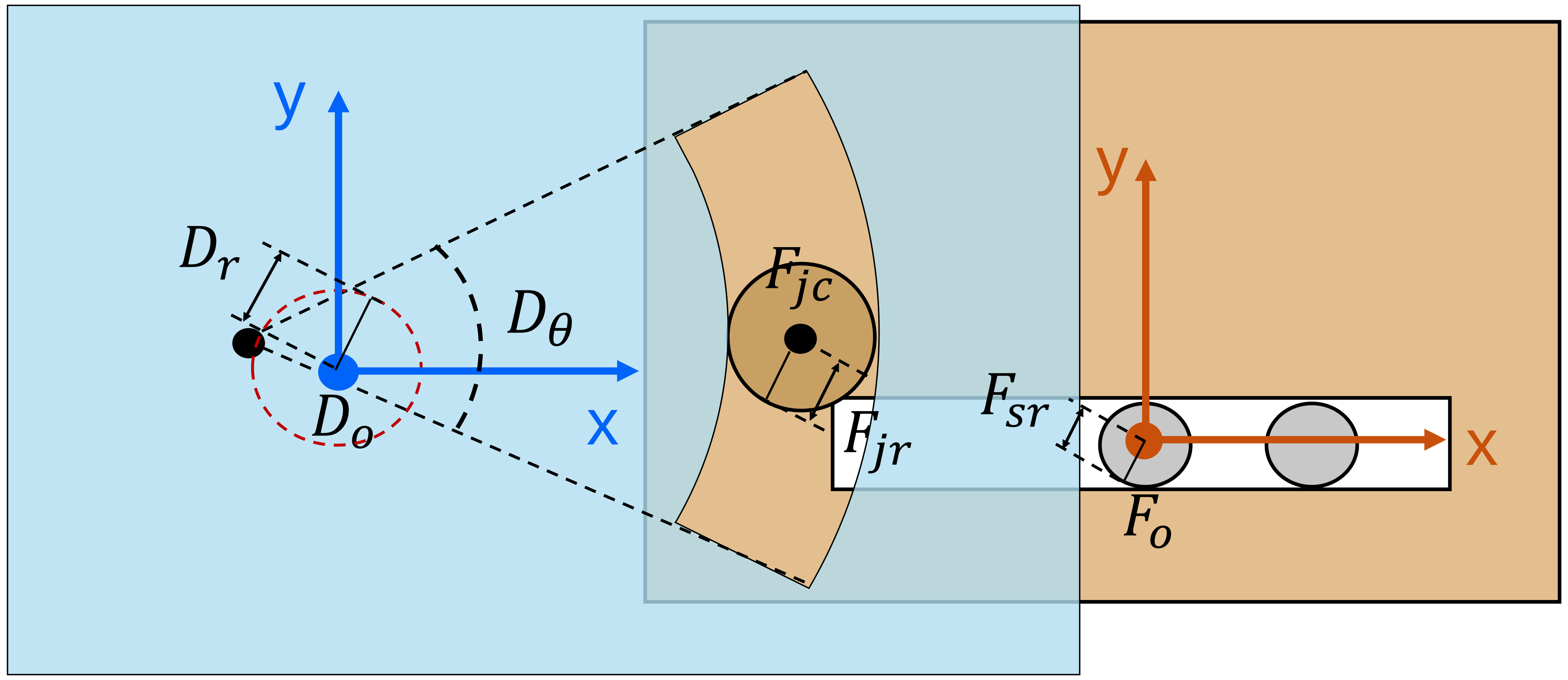


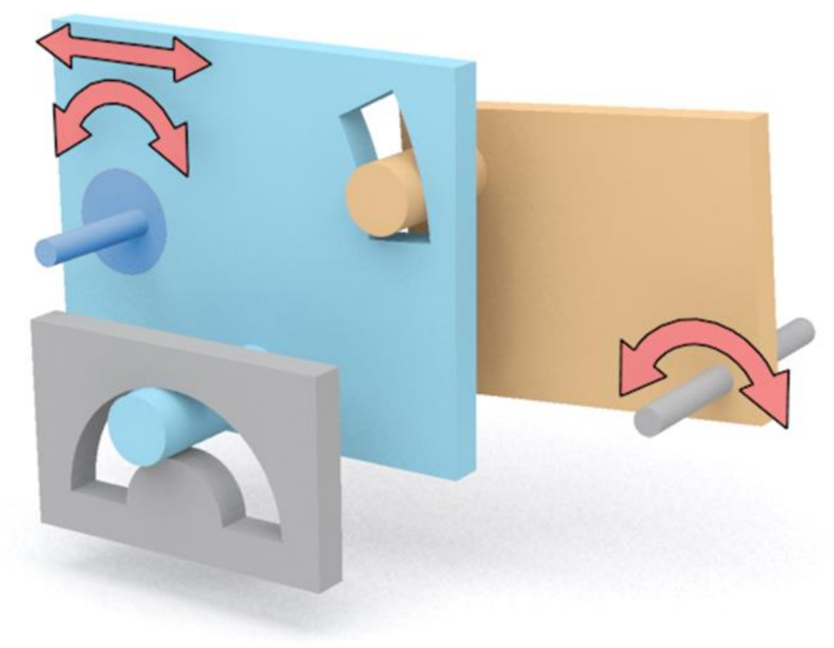


#10 $O_zT \rightarrow T_x$

- D_o: (0.00, 0.00, 0.00)
- D_θ: $(-\frac{\pi}{9}, \frac{\pi}{9})$

- F_o: (1.20, -0.20, -0.30)
- F_{jc}: (-0.40, 0.30)
- F_{jr}: 0.075
- F_{sr}: 0.035

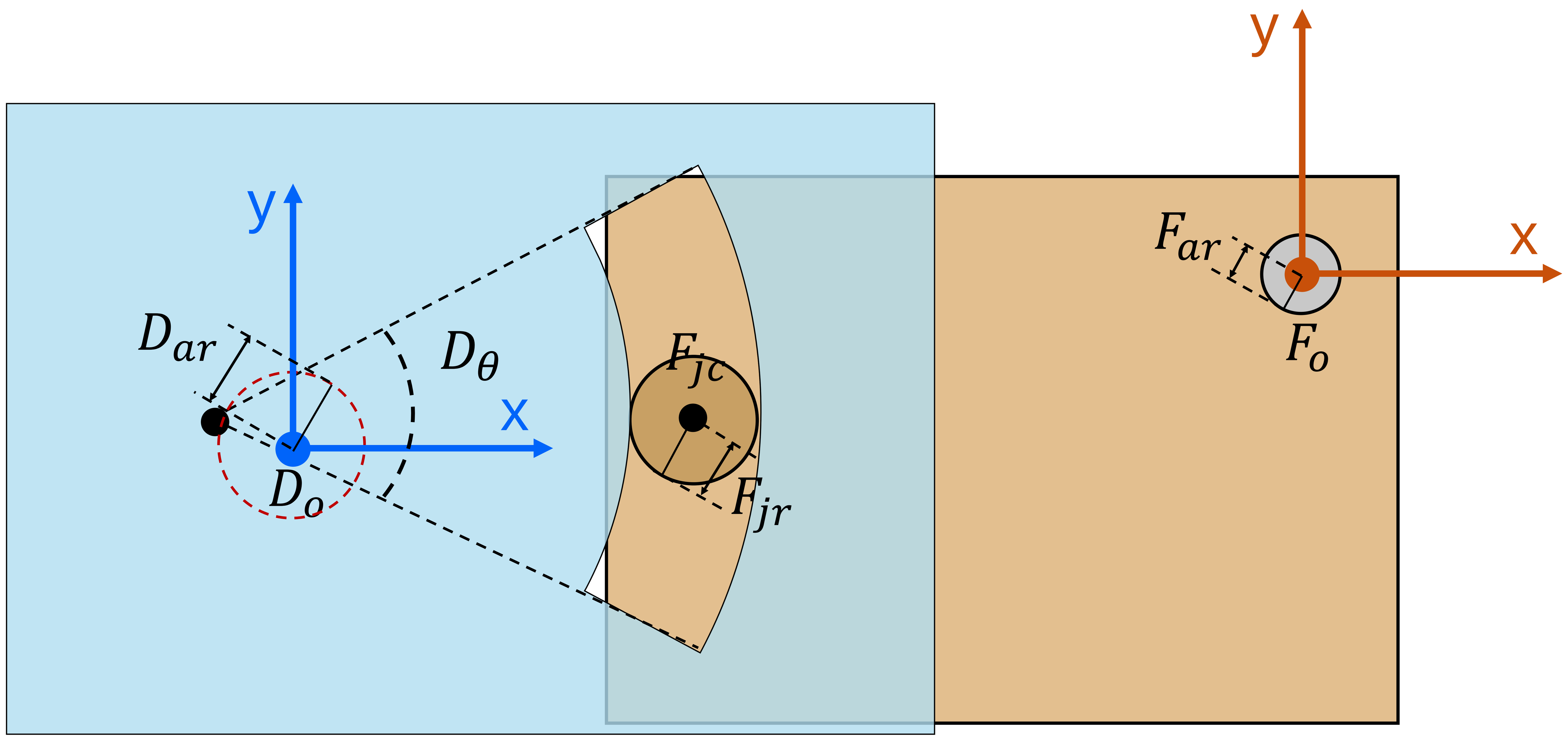




#11 $O_zT \rightarrow O_z$

- D_o: (0.00, 0.00, 0.00)
- D_θ: $(-\frac{\pi}{9}, \frac{\pi}{9})$

- F_o: (0.90, 0.30, -0.30)
- F_{jc}: (-0.40, -0.30)
- F_{jr}: 0.075
- F_{ar}: 0.03



Part 3:
Failure Cases of Kinematics Computation

Overview

Among the 11 eleMechs, only 3 of them have failure cases when computing kinematics. All these 3 eleMechs require computing intersection points between two geometries, which may not be always satisfied.

#3 $\mathbf{R}_z \rightarrow \mathbf{O}_z\mathbf{T}$

#5 $\mathbf{O}_z \rightarrow \mathbf{O}_z$

#11 $\mathbf{O}_z\mathbf{T} \rightarrow \mathbf{O}_z$

#3 $R_z \rightarrow O_z T$

When the distance between m and s is larger than $R + r$, $R_z \rightarrow O_z T$ fails, because the circle centered at m and the circle centered at s do not have any intersection points.

$$\begin{cases} (x - m_x)^2 + (y - m_y)^2 = R^2 \\ (x - s_x)^2 + (y - s_y)^2 = r^2 \end{cases}$$

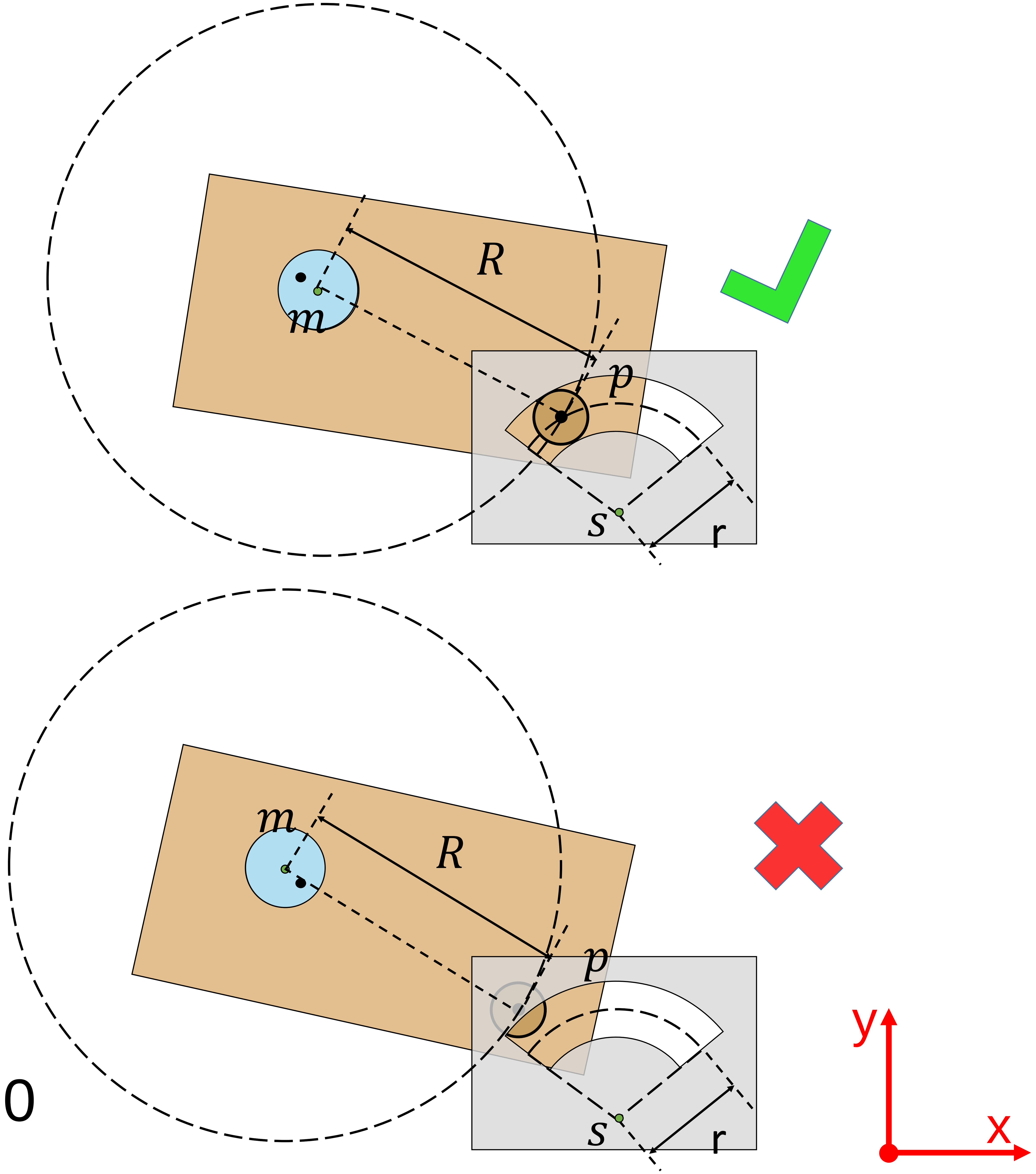
Let p be the intersection point (if any), then

$$p_y = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + s_y$$

The failure cases happen when

$$b^2 - 4ac < 0$$

Note: equations to compute a, b, and c are on page 10



#5 $O_z \rightarrow O_z$

When line l (green line in the right figures) has no intersection with the circle centered at s, $O_z \rightarrow O_z$ fails.

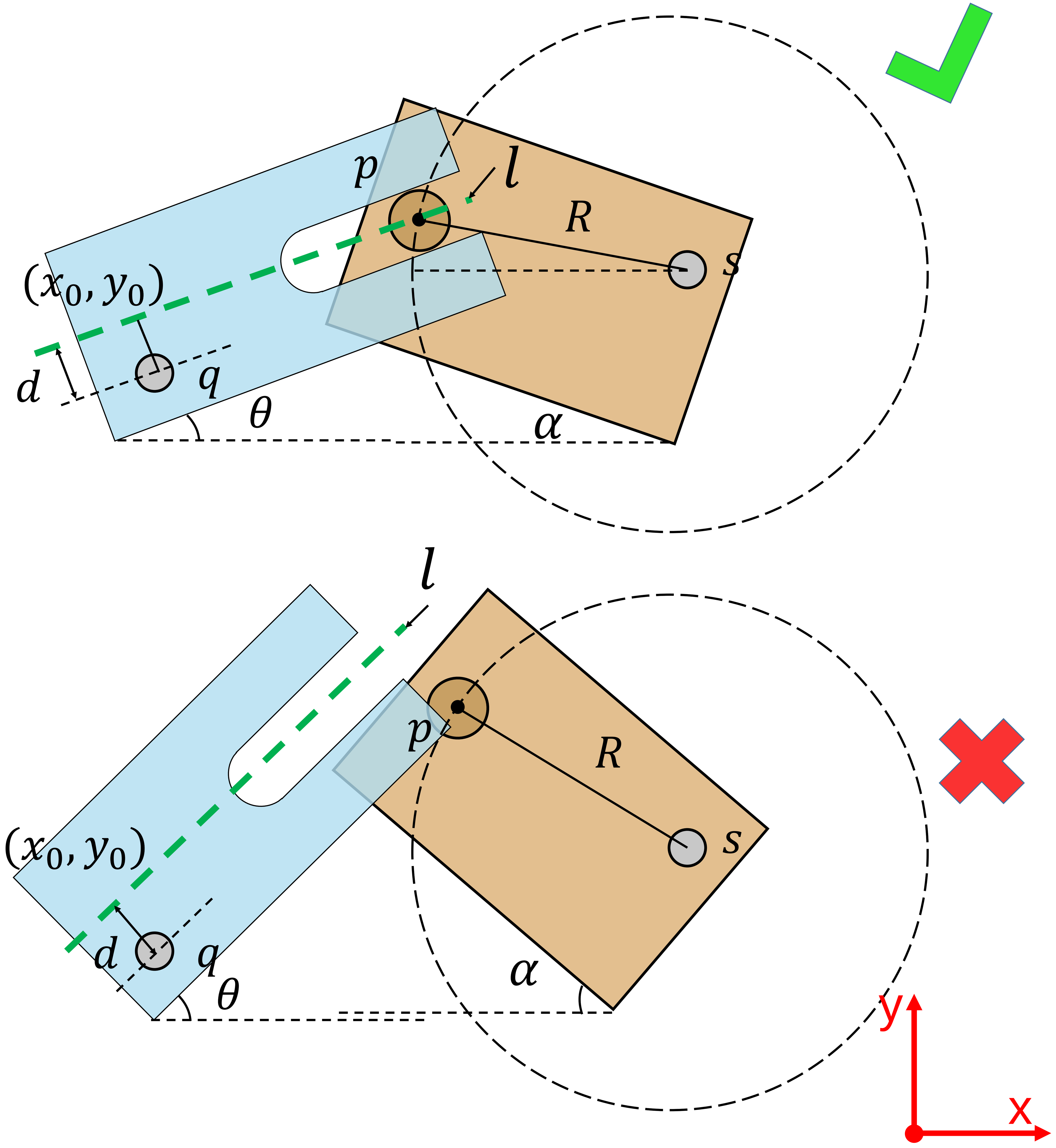
Let p be the intersection point (if any), then

$$p_x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The failure cases happen when

$$b^2 - 4ac < 0$$

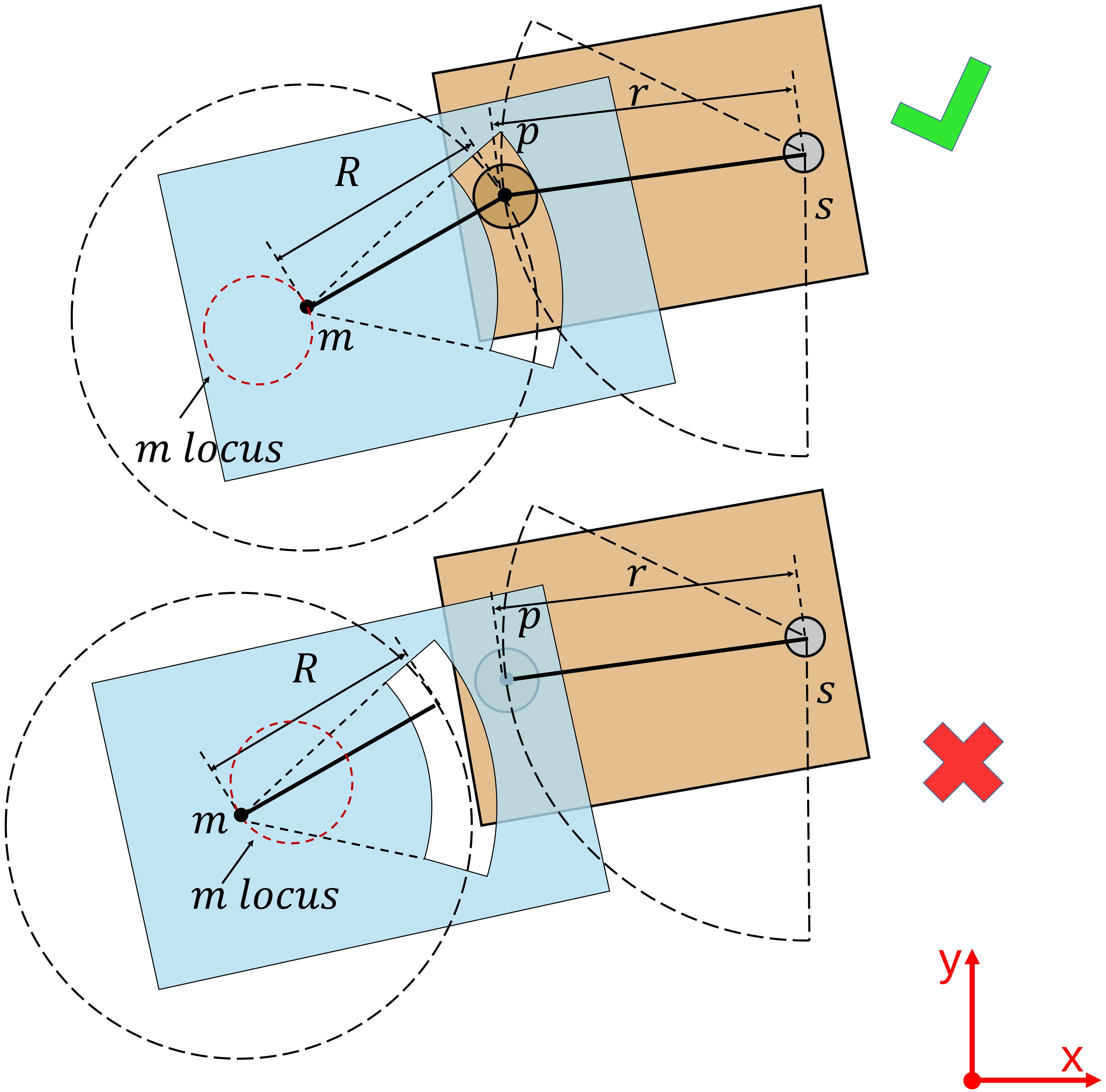
Note: equations to compute a, b, and c are on page 14



#11 $O_zT \rightarrow O_z$

When the distance between m and s is larger than $R + r$, $O_zT \rightarrow O_z$ fails.

This failure case is similar to that of #3 $R_z \rightarrow O_zT$



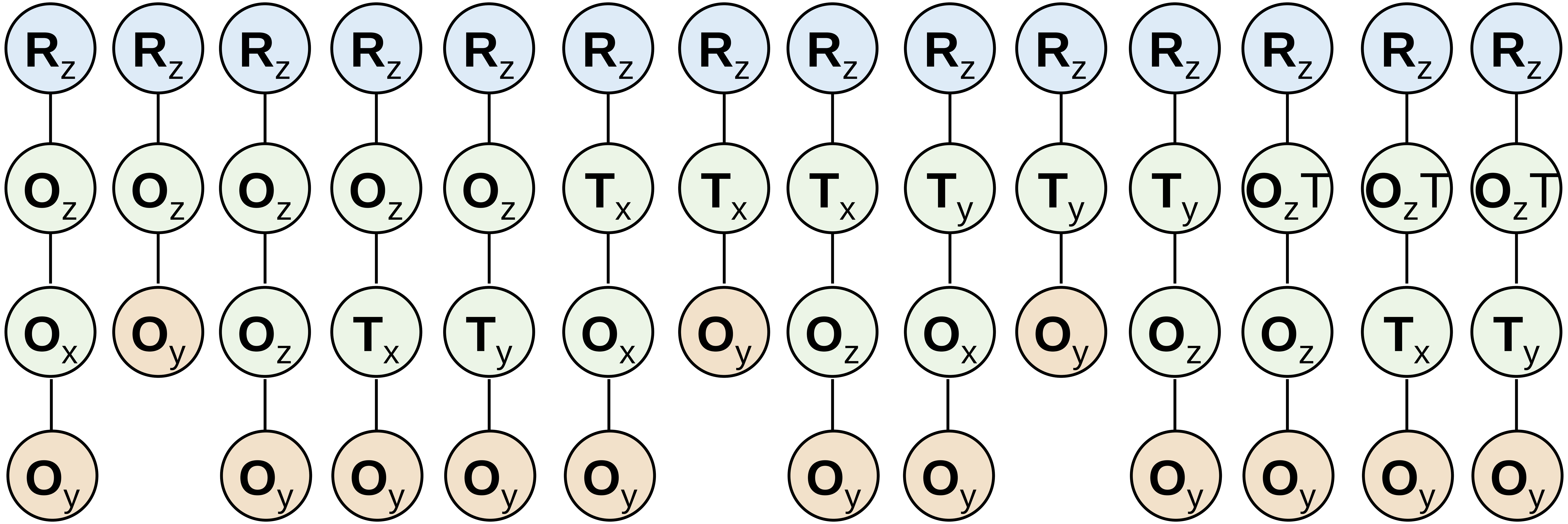
Part 4:
All Possible Motion Transfer Chains
(length ≤ 3)

End-effector Motion: $\mathbf{O}_x$

- Number of Chains: **17**

End-effector Motion: $\mathbf{O}_y$

- Number of Chains: **14**

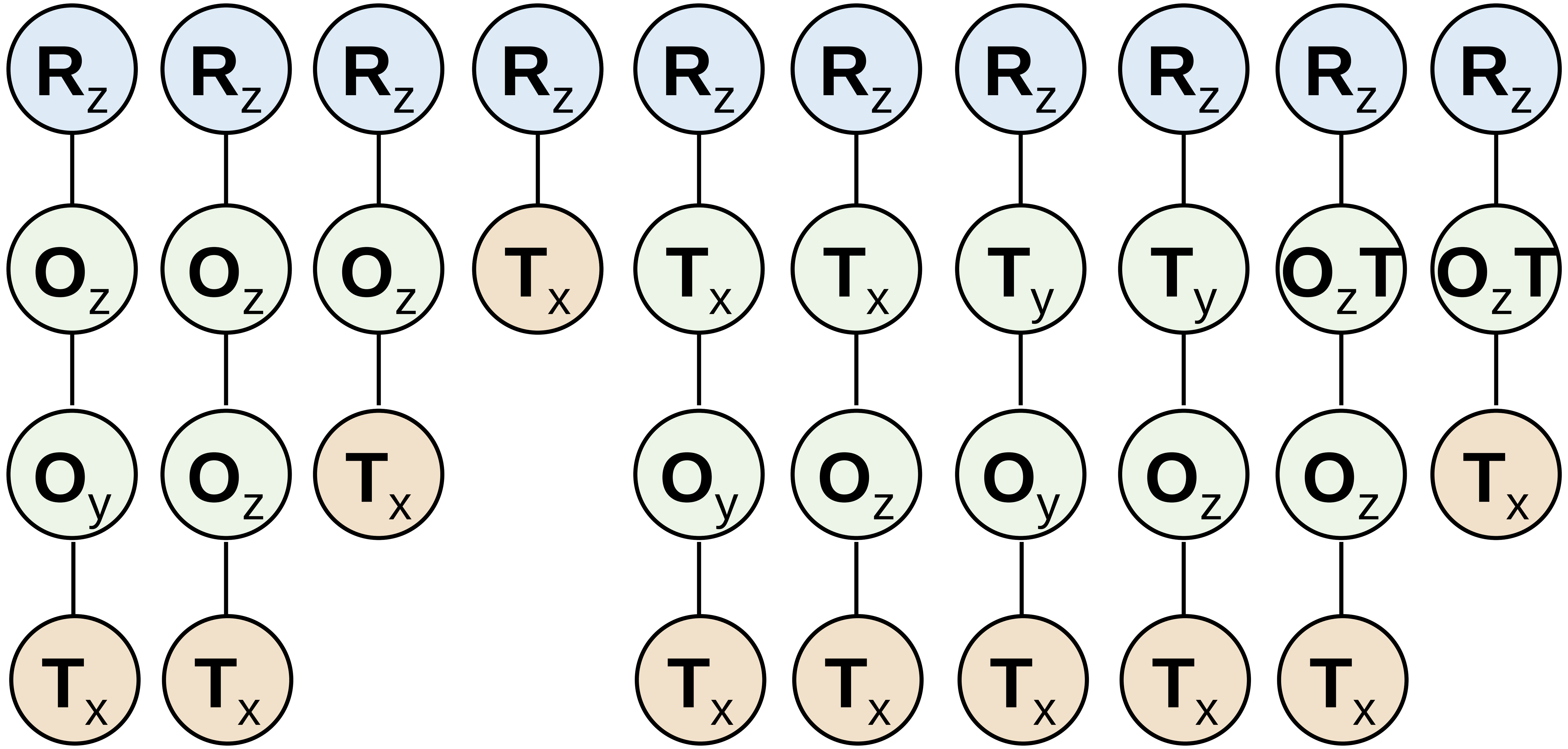


End-effector Motion: $\mathbf{O}_z$

- Number of Chains: **19**

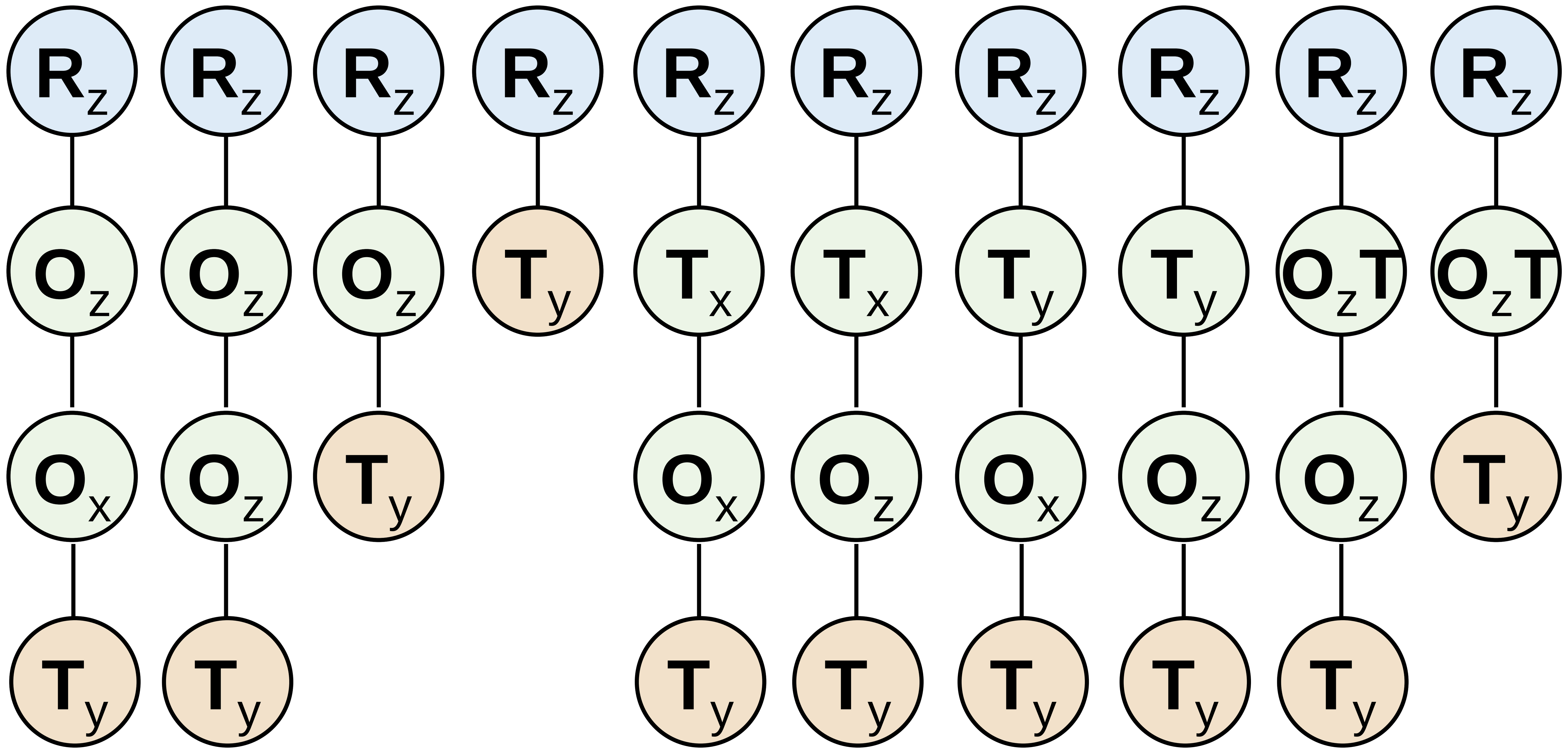
End-effector Motion: $\mathbf{T}_x$

- Number of Chains: **10**



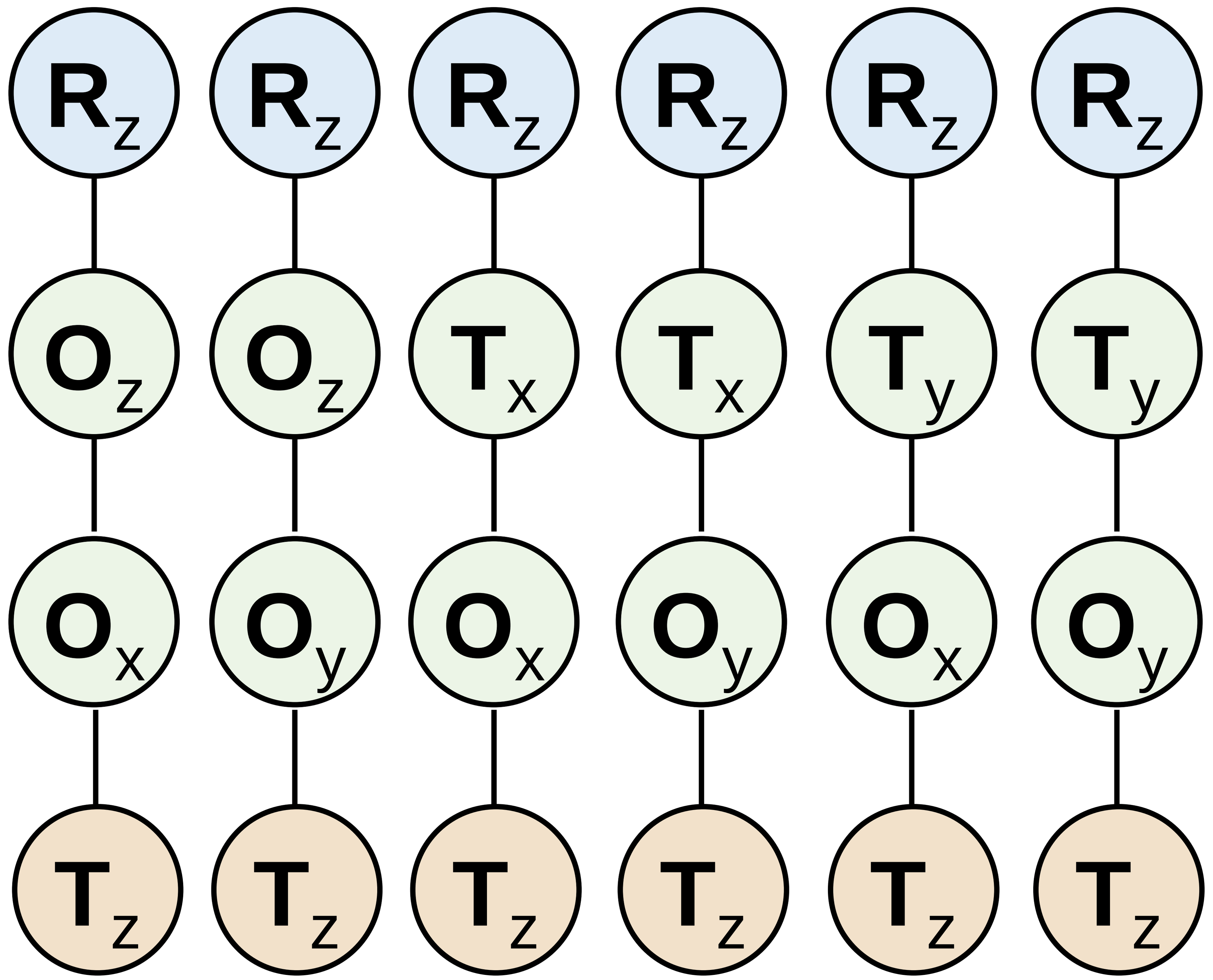
End-effector Motion: $\mathbf{T}_y$

- Number of Chains: **10**



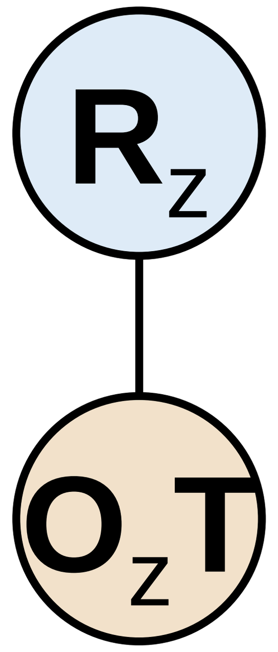
End-effector Motion: $\mathbf{T}_z$

- Number of Chains: **6**



End-effector Motion: $\mathbf{O_z T}$

- Number of Chains: **1**



The End