Supplementary Material for *Exact 3D Path Generation via 3D Cam-Linkage Mechanisms*

In this supplementary material, we first provide technical details about topologies for our 3D cam-linkage mechanism. Next, we provide equations of the input parametric curves/motions that are used in the paper. Third, we present technical details about the experiment of generating a 3D path with arbitrary shape. Lastly, we show all the 20 input curves and corresponding mechanisms in the experiment of generating 3D paths with varying shape complexity.

1 Topologies of Cam-Linkage Mechanism

Choosing 5-bar instead of 4-bar spatial linkage. In the followings, we explain why we do not choose a 4-bar spatial linkage for our 3D cam-linkage mechanism. Assume that we combine a 4-bar spatial linkage and two 3D cams using the same method in Section 4 of the paper. Then, we have the following equation according to Equation 1 in the paper:

$$f_2 + f_3 = 6 \tag{1}$$

Since our link joints support at most 3 DOFs, both joints J_2 and J_3 have to be a spherical joint. However, a bar with two spherical joints at its ends will have 1-DOF uncontrollable motion (i.e., 1-DOF rotation around the bar axis), making the linkage unable to output arbitrary 3-DOF motion. Due to this reason, we choose a 5-bar instead of a 4-bar spatial linkage for composing our 3D cam-linkage mechanism.

128 topologies of our cam-linkage mechanism. In the paper, we have introduced that there are 128 different topologies of the 3D cam-linkage mechanism. Below, we show how we derive this total number of topologies for the mechanism, i.e., the total number of combinations of the five link joints.

The total number of combinations for the active joints J_1 and J_5 is $2 \times 2 = 4$ since J_1 can be $\{C, U\}$ while J_5 can be $\{R, P\}$. The total number of combinations for the passive joints J_2, J_3, and J_4 that satisfy Equation 2 in the paper can be classified into the following two classes:

- Class 1: $f_2 = 2,\ f_3 = 2,\ f_4 = 2$. The number of combinations for this class is $2 \times 2 \times 2 = 8$ since f_2 can be $\{C, U\}$, f_3 can be $\{C, U\}$, and f_4 can be $\{C, U\}$.
- Class 2: $f_a = 1,\ f_b = 2,\ f_c = 3$. Let's assume $f_a = f_2$, $f_b = f_3$, and $f_c = f_4$. The number of combinations for this case is $2 \times 2 \times 1 = 4$ since f_2 can be $\{R, P\}$, f_3 can be $\{C, U\}$, and f_4 can be $\{S\}$. Moreover, we have $3! = 6$ permutations of assigning the number of DOFs (i.e., 1, 2, 3) to the three passive joints (i.e., J_2, J_3, and J_4). Hence, the number of combinations for this class is $4 \times 6 = 24$.

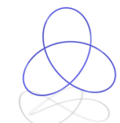
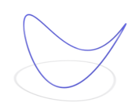
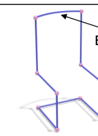
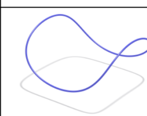
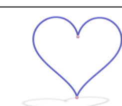
Thus, the total number of combinations for the passive joints is $8 + 24 = 32$. Overall, the total number of combinations of the five link joints is $4 \times 32 = 128$.

2 Equations of Input Parametric Curves

We used parametric curves as input space curves for designing 3D cam-linkage mechanisms in the paper. We provide equations of these parametric curves in Table 1.

We also provide the equation of the input motion for the generalized mechanism in Figure 15 of the paper. Denote $\mathbf{M}(\mathbf{R}, \mathbf{T})$ as a rigid transformation matrix with rotation $\mathbf{R}$ and translation $\mathbf{T}$. We represent the rotation $\mathbf{R}$ with Euler angle $\{\alpha, \beta, \gamma\}$, i.e., $\mathbf{R} = \mathbf{R}(\alpha, \beta, \gamma)$, where

Table 1: *Equations of input parametric curves.*

Fig	Name	Equation	Curve
1	Trefoil Knot	$\begin{cases} x = \cos(t) + 2\cos(2t) \\ y = \sin(t) - 2\sin(2t) \\ z = 1.2\sin(3t) \end{cases}$	
7	Pancake	$\begin{cases} x = 2\cos(t) \\ y = 2\sin(t) \\ z = \cos(2t) \end{cases}$	
8a	Bézier curve	$\begin{cases} x = \frac{3\sqrt{3}}{2}t^2(1-t) + \frac{\sqrt{3}}{6}t^3 \\ y = 3t\,(1-t)^2 + 1.5t^2(1-t) + \frac{\sqrt{3}}{6}t^3 \\ z = \frac{\sqrt{6}}{3}t^3 \end{cases}$	
8b	Chair	Polyline vertices: {(0, 0, 10), (0, 0, 4), (0, 5, 4), (0, 5, 0), (0, 0, 0), (5, 0, 0), (5, 5, 0), (5, 5, 4), (5, 0, 4), (5, 0, 10)} Bézier curve control points: {(5, 0, 10), (2.5, −1.4, 10), (0, 0, 10)}	
8d	Conical Rose	$\begin{cases} x = 2\cos(2t)\cos(t) \\ y = 2\cos(2t)\sin(t) \\ z = 1.4\cos(2t) \end{cases}$	
9	Hyperbolic Boundary	$\begin{cases} x = 2(\cos(2.5)\cos(t)\cos(2t) - \sin(t)\sin(2t)) \\ y = 2(\cos(2.5)\sin(t)\cos(2t) + \cos(t)\sin(2t)) \\ z = 1.2\sin(2.5)\cos(2t) \end{cases}$	
12	Tennis Ball Seam	$\begin{cases} x = 2\cos(t) + \cos(3t) \\ y = 2\sin(t) - \sin(3t) \\ z = 1.2\sin(2t) \end{cases}$	
13	Heart	$\begin{cases} x = 16\sin(t)^3 \\ y = 13\cos(t) - 5\cos(2t) \\ \quad -2\cos(3t) - \cos(4t) \end{cases}$	

$$\alpha = \begin{cases} \frac{\pi}{2}, & t \in [0, 0.12\pi); \\ \frac{0.5}{0.78}(0.9\pi - t), & t \in [0.12\pi, 0.9\pi); \\ 0, & t \in [0.9\pi, \pi], \end{cases} \tag{2}$$

$$\beta = 0, \quad t \in [0, \pi], \tag{3}$$

$$\gamma = \begin{cases} \frac{\pi}{2}, & t \in [0, 0.33\pi); \\ \frac{0.5}{0.57}(0.9\pi - t), & t \in [0.33\pi, 0.9\pi); \\ 0, & t \in [0.9\pi, \pi]. \end{cases} \quad (4)$$

We represent the translation $\mathbf{T}$ with a translation vector $(x, y, z)^T$, where

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R} \times ((1 + \cos(t)) \begin{bmatrix} 10 \\ 15 \\ -20 \end{bmatrix} + \begin{bmatrix} 31 \\ -17 \\ 23 \end{bmatrix}) + \begin{bmatrix} 40 \\ 70 \\ 60 \end{bmatrix}, \quad t \in [0, \pi], \quad (5)$$

3 Generating a 3D path with Arbitrary Shape

We first explain how to choose the geometry of a 3-DOF 5-bar spatial linkage such that the linkage's output motion space has a continuous and volumetric portion $\mathbf{\Phi}$ that is singularity-free and collision-free. Next, we explain how to model a continuous 3D curve with arbitrary shape and a finite number of C^0 points by moving a particle with random acceleration in a fixed sphere.

Computing a 3-DOF spatial linkage's motion space. Without loss of generality, we assume two conditions for a given 3-DOF 5-bar spatial linkage: 1) the Jacobian Matrix $\mathbf{A}$ of the linkage's forward kinematics is positive, i.e., $\sigma_{\text{min}}(\mathbf{A}) \geq \mu$; and 2) the initial state of the spatial linkage is singularity-free and collision-free. We show that there always exists a continuous and volumetric portion $\mathbf{\Phi}$ that is singularity-free and collision-free in the output motion space of a linkage that satisfies the two conditions.

According to the kinematic modeling in the paper, the motion transfer function $\mathbf{f}(\cdot)$ of the 3-DOF linkage can be expressed as $\mathbf{B}(t) = \mathbf{f}([\mathbf{s}_1, \mathbf{s}_5])$, where the active joint angles $[\mathbf{s}_1, \mathbf{s}_5]$ represent the input motion while the end-effector point position $\mathbf{B}(t)$ represents the output motion. Denote the input motion space as $\mathbf{\Omega}_{\text{in}}$ and the output motion space as $\mathbf{\Omega}_{\text{out}}$. For a certain point $\mathbf{x} \in \mathbf{\Omega}_{\text{in}}$, there exists an open ball with radius r covering $\mathbf{x}$, belonging to $\mathbf{\Omega}_{\text{in}}$, i.e. $\exists U_r(\mathbf{x}) \subset \mathbf{\Omega}_{\text{in}}$. This statement is true since the input motion space $\mathbf{\Omega}_{\text{in}}$ is the tensor product of the continuous space of each active joint angle. Due to the continuity of the motion transfer function $\mathbf{f}(\cdot)$ and its inverse, there exists an open ball with radius R in the output motion space $\mathbf{\Omega}_{\text{out}}$ that covers $\mathbf{f}(\mathbf{x})$, i.e., $\exists U_R(\mathbf{f}(\mathbf{x})) \subset \mathbf{\Omega}_{\text{out}}$ [Rudin 1991]. The radius R of the open ball in the output motion space should be at least μr due to the positive Jacobian Matrix $\mathbf{A}$, i.e., $\sigma_{\text{min}}(\mathbf{A}) \geq \mu$.

Since the initial state of the spatial linkage is singularity-free and collision-free, we can choose the active joint angles of the initial state as $\mathbf{x} \in \mathbf{\Omega}_{\text{in}}$, and compute the open ball $U_R(\mathbf{f}(\mathbf{x})) \subset \mathbf{\Omega}_{\text{out}}$. It is obvious that $U_R(\mathbf{f}(\mathbf{x}))$ is a continuous and volumetric portion $\mathbf{\Phi}$ that is singularity-free and collision-free in the output motion space $\mathbf{\Omega}_{\text{out}}$ of the linkage. To further increase the size of the portion $\mathbf{\Phi}$, we can expand it by using many open balls in the output motion space that are overlapped with one another without any gap. This is achieved by densely sampling a number of points in the input motion space $\mathbf{\Omega}_{\text{in}}$, computing the corresponding open balls in the output motion space, and choosing those open balls that are singularity-free and collision-free and can expand the portion $\mathbf{\Phi}$ while maintaining its continuity.

Modeling a 3D curve with arbitrary shape. We model a 3D curve with arbitrary shape by moving a particle randomly in a fixed sphere with radius R. In detail, we initialize the position $\mathbf{p}_0$ of the particle as the center of the sphere and initialize the velocity $\mathbf{v}_0$ of the particle randomly. Next, we apply some random force $\mathbf{F}_r(t)$ on the particle, where each component of the force vector is a randomly generated number within a prescribed range. To avoid this particle to go out of the sphere, we model an obstacle energy $E_{\text{obst}}(d) = (\ln(d/d_1))^2$, where d is the distance to the sphere and $d_1 = R/2$. The obstacle energy $E_{\text{obst}}(d)$ is infinite at $d = 0$, and reduces to 0 at $d = d_1$. To avoid the particle to move too fast, we model a resistive force, i.e., $\mathbf{F}_\mathbf{v} = -k\|\mathbf{v}\|\mathbf{v}$, where k is a positive coefficient and set as 1 in our experiment. Hence, the total force applied on the particle is:

$$\mathbf{F}(t) = \mathbf{F}_r(t) - \delta_{d<d_1} \nabla E_{\text{obst}} - k\|\mathbf{v}\|\mathbf{v} \quad (6)$$

The particle's acceleration at time t is $\mathbf{a}(t) = \mathbf{F}(t)/m$, where m is the mass of the particle. The particle's velocity $\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(t_1)dt_1$ is a continues function. And the particle's position $\mathbf{p}(t) = \mathbf{p}_0 + \int_0^t \mathbf{v}(t_1)dt_1$ forms a C^1 curve. To introduce a C^0 point in the C^1 curve at time t, we replace the particle's velocity $\mathbf{v}(t)$ calculated from the integration with a randomly generated velocity vector that has a different direction from $\mathbf{v}(t)$. We introduce a finite number of C^0 points in the C^1 curve by applying the velocity modification operation multiple times.

4 Generating 3D Paths with Varying Shape Complexity

In Figure 1, we show all the 20 input 3D curves with increasing shape complexity as well as the corresponding path generation mechanisms designed by our approach.

All the 20 input curves $\{\mathbf{R}_k\}$, $1 \leq k \leq 20$ are continuous and closed with increasing shape complexity, where Length$(\mathbf{R}_{k+1}) = 1.1 \cdot$ Length$(\mathbf{R}_k)$ and $\mathbf{R}_k$ has $k - 1$ C^0 points. To generate each closed input curve $\mathbf{R}_k$, we first use the particle-based method mentioned above to generate an open curve with $k - 1$ C^0 points and then make the curve closed (without introducing new C^0 points) by connecting the starting point and ending point using a Hermite curve.

References

RUDIN, W. 1991. *Functional Analysis*, 2nd ed. McGraw-Hill.

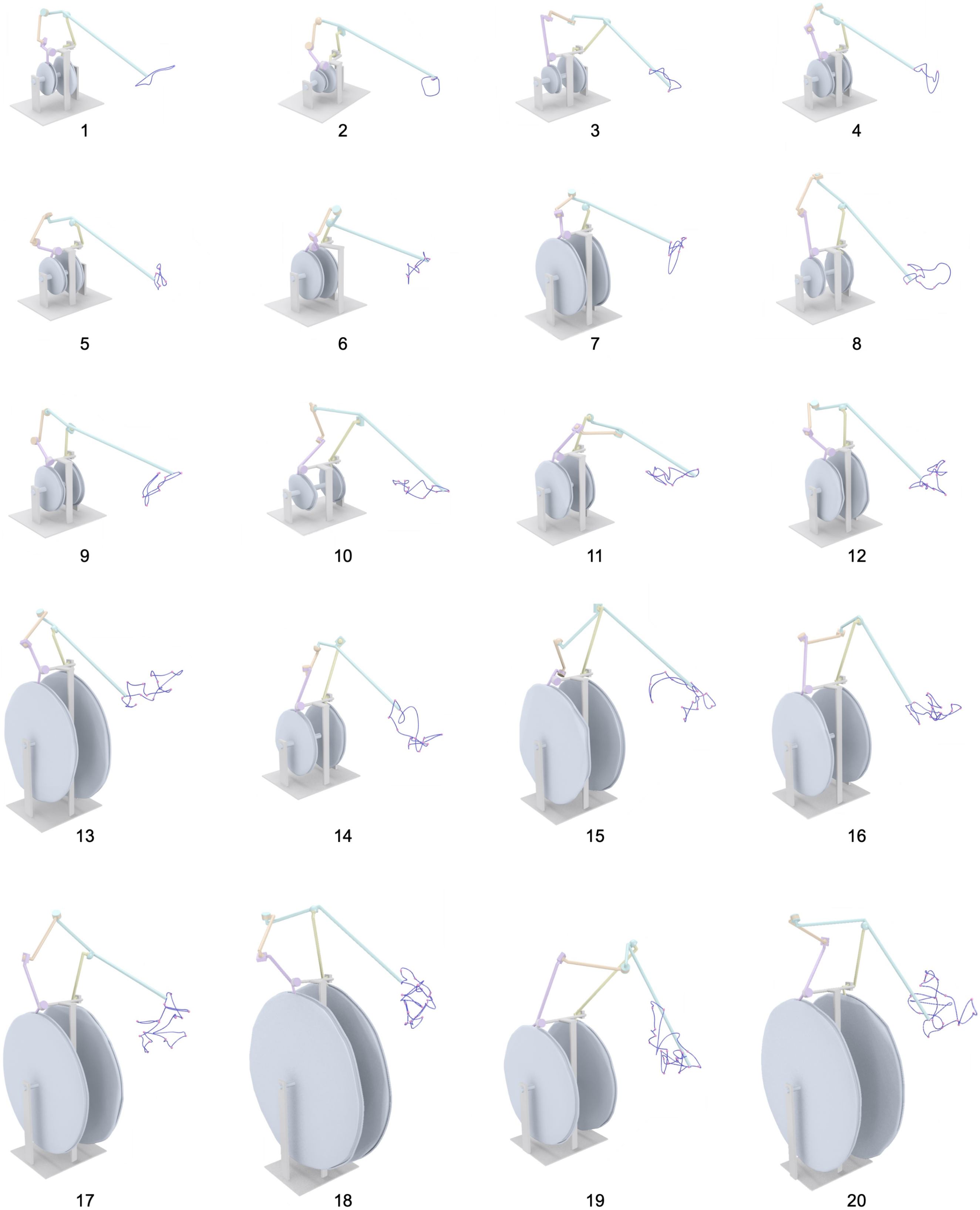


Figure 1: *Our mechanisms for generating input 3D curves with increasing shape complexity, where the curve index k is indicated by the number beside each curve.*