Supplementary Material for mpcMech: Multi-Point Conjugation Mechanisms

This supplementary material is composed of four parts. The first part presents a proof to the equivalence of the dynamic form closure condition (see Section 4.3 in the paper). The second part shows the process to prove the kinematic relation about relative velocity from the coincident contact point condition (see Section 5.1 in the paper). The third part provides details about how to avoid the case that the generalized normal is perpendicular to the follower's motion space (see Section 5.2 in the paper). The last part provides details of our optimization solver (see Section 5.2 in the paper).

1 Dynamic Form Closure Condition

In the paper we present the condition of the dynamic form closure:

$$U \subset \bigcup_{k=1}^{K} U_k(t), \quad \forall t \in [0,T]. \tag{1}$$

This condition can be understood as a requirement of covering a $(N-1)$-dimensional unit hypersphere U by K 5-dimensional unit semi-hypersphere $\{U_k(t)\}_{1\leqslant k\leqslant K}$ with the corresponding generalized normal $\{\hat{\mathbf{n}}_k(t)\}_{1\leqslant k\leqslant K}$. If $\hat{\mathbf{n}}_k(t) \not\perp U$, then we can define $\tilde{\mathbf{n}}_k(t)$, which is a normalized vector that represents a projection of the generalized normal $\hat{\mathbf{n}}_k(t)$ onto the linear subspace spanned by U. Thus the equation 1 can be again viewed as a requirement of covering a $(N-1)$-dimensional unit hypersphere U by K $(N-1)$-dimensional unit semi-hypersphere with center point $-\tilde{\mathbf{n}}_k(t)$. In the paper we claim that equation 1 is equivalent to:

$$\mathbf{0} \in \text{interior}\left(\text{conv}(\{\tilde{\mathbf{n}}_k(t)\}_{1\leqslant k\leqslant K})\right), \quad \forall t \in [0,T]. \tag{2}$$

This equivalence can be easily derived from the following proposition:

Proposition 1 $\mathbf{S}^n = \{\mathbf{u} \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\| = 1\}$ is a n-dimensional unit hypersphere. $\{\mathbf{H}_k\}_{1\leqslant k\leqslant K}$ is a set of K n-dimensional unit semi-hypersphere with center point $\mathbf{s}_k$, namely, $\mathbf{H}_k = \{\mathbf{u} \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\| = 1, \mathbf{u}\cdot\mathbf{s}_k > 0\}$. $\mathbf{S}^n$ is covered by these K semi-hypersphere, that is,

$$\mathbf{S}^n \subset \bigcup_{k=1}^{K} H_k,$$

if and only if $\mathbf{0} \in \text{interior}\left(\text{conv}(\{\mathbf{s}_k\}_{1\leqslant k\leqslant K})\right)$.

Proof 1 ($\Leftarrow$) $\exists$ a hyperball $B_r(\mathbf{0}) \subset \text{conv}(\{\mathbf{s}_k\}_{k=1}^K)$, then for $\forall \mathbf{u} \in \mathbf{S}^n$, $\|\mathbf{u}\| = 1$, we have $\frac{r}{2}\mathbf{u} \in B_r(\mathbf{0})$, thus $\frac{r}{2}\mathbf{u} = \sum_{k=1}^{K} \lambda_k \mathbf{s}_k$, $\sum_{k=1}^{K} \lambda_k = 1, \lambda_k \geqslant 0$. Then

$$\frac{r}{2}\|\mathbf{u}\|^2 = \sum_{k=1}^{K} \lambda_k (\mathbf{s}_k \cdot \mathbf{u}).$$

Assume $\mathbf{s}_k \cdot \mathbf{u} \leqslant 0$, for $\forall 1 \leqslant k \leqslant K$, then the right of the above equation is nonnegative, but the left is positive, contradiction. Therefore $\exists 1 \leqslant k \leqslant K$, $\mathbf{s}_k \cdot \mathbf{u} > 0$, that means $\mathbf{S}^n$ is covered by these K semi-hypersphere.

($\Rightarrow$) $\text{conv}(\{\mathbf{s}_k\}_{1\leqslant k\leqslant K})$ is convex, so its interior set $\text{int}\left(\text{conv}(\{\mathbf{s}_k\}_{1\leqslant k\leqslant K})\right)$ is convex. We denote its interior set as $\mathbf{I}$.

Assume that $\mathbf{0} \notin \mathbf{I}$, according to Separating Hyperplane Theorem, there exists a hyperplane $\mathbf{a}^\top\mathbf{x} + b = 0$ separate $\mathbf{0}$ and $\mathbf{I}$, that is,

$$\begin{cases} \mathbf{a}^\top\mathbf{0} + b \geqslant 0, \\ \mathbf{a}^\top\mathbf{x} + b \leqslant 0, \forall \mathbf{x} \in \mathbf{I}, \end{cases}$$

Thus $\forall \mathbf{x} \in \mathbf{I}$, we have $\mathbf{a}^\top\mathbf{x} \leqslant -b \leqslant 0$. Then it can be easily known that $\mathbf{a}^\top\mathbf{x} \leqslant 0$, $\forall \mathbf{x} \in \text{conv}(\{\mathbf{s}_k\}_{1\leqslant k\leqslant K})$. Set $\mathbf{x} = \mathbf{s}_k, 1 \leqslant k \leqslant K$, we have $\mathbf{a}^\top\mathbf{s}_k \leqslant 0$, so $\frac{\mathbf{a}}{\|\mathbf{a}\|} \in \mathbf{S}^n$ is not covered by each $\mathbf{H}_k$, contradiction.

2 Relative Velocity Condition for Point Conjugation

In the paper, we have the coincident contact point condition:

$$\mathbf{F}_1^2\mathbf{M}^1(t)\mathbf{c}_k^1(t) = \mathbf{p}_k(t) = \mathbf{M}^2(t)\mathbf{c}_k^2(t) \tag{3}$$

where $\mathbf{c}_k^1(t)$ and $\mathbf{c}_k^2(t)$ are the conjugate curve pair in the local frame, $\mathbf{M}^1(t)$ and $\mathbf{M}^2(t)$ are their respective motion, and $\mathbf{F}_1^2$ is the transformation (i.e., a 3D translation) of surface $\mathbf{S}^1$'s local frame to surface $\mathbf{S}^2$'s local frame. Taking derivatives of Equation 3, we have

$$\mathbf{F}_1^2\dot{\mathbf{M}}^1(t)\mathbf{c}_k^1(t) + \mathbf{F}_1^2\mathbf{M}^1(t)\dot{\mathbf{c}}_k^1(t) = \dot{\mathbf{M}}^2(t)\mathbf{c}_k^2(t) + \mathbf{M}^2(t)\dot{\mathbf{c}}_k^2(t), \tag{4}$$

namely,

$$\mathbf{v}_k^1(t) + \mathbf{T}_k^1(t) = \mathbf{v}_k^2(t) + \mathbf{T}_k^2(t), \tag{5}$$

where $\mathbf{v}_k^\alpha(t)$ is the velocity at the conjugation point of $\mathbf{c}_k^\alpha(t)$ in the global frame and $\mathbf{T}_k^\alpha(t)$ is the tangent vector at the conjugation point of $\mathbf{c}_k^\alpha(t)$ in the global frame. Therefore, the relative velocity $\mathbf{v}_k^{12}(t) := \mathbf{v}_k^1(t) - \mathbf{v}_k^2(t) = \mathbf{T}_k^2(t) - \mathbf{T}_k^1(t)$.

3 Constraint of Avoiding $\hat{\mathbf{n}}_k(t) \not\perp U$

From our theory of dynamic form closure in the paper, we know that if the generalized normal $\hat{\mathbf{n}}_k(t)$ is perpendicular to the motion set U, the kth conjugate curve pair fails to contribute to form close the follower surface at time t. So we have to take strategy in our optimziation solver to avoid this case. And In our experiments we find that we need to take different methods for different type of the follower suppoert joint to obtain good results. We focus on 3 types metioned in the paper: mpcMech_3R, mpcMech_1R and mpcMech_1R1T.

For mpcMech_3R, as mentioned in the paper, $\tilde{\mathbf{n}}_k(t) = \frac{\mathbf{p}_k(t) \times \mathbf{n}_k(t)}{\|\mathbf{p}_k(t) \times \mathbf{n}_k(t)\|}$, we need to introduce a constraint to avoid $\mathbf{p}_k(t) \times \mathbf{n}_k(t) = \mathbf{0}$:

$$C_{\text{closure\_3R}} = \left(\mathbf{n}_k(t) \cdot \frac{\mathbf{p}_k(t)}{\|\mathbf{p}_k(t)\|}\right)^2 - \cos^2(\gamma_0 + \frac{\pi}{2}) \leqslant 0, \tag{6}$$

where γ_0 is a threshold set as $10°$ in our experiments.

For mpcMech_1R, we assume the motion set $U = \{(0,0,0,0,0,\pm 1)^\top\}$. And instead of adding a constraint like the mpcMech_3R do, we minimize:

$$E_{\text{closure\_1R}} = -\sum_i \left(\frac{\mathbf{V} \times \mathbf{p}_k(t_i)}{\|\mathbf{V} \times \mathbf{p}_k(t_i)\|} \cdot \mathbf{n}_k(t_i) \right)^2, \tag{7}$$

where $\mathbf{V} = (0,0,1)^\top$.

For mpcMech_1R1T, we assume the motion space $U = \{(0,0,t,0,0,\omega)^\top \mid t^2 + \omega^2 = 1\}$. Similar to mpcMech_1R , we minimize:

$$E_{\text{closure\_1R1T}} = E_{\text{closure\_1R}} - \sum_i (\mathbf{V} \cdot \mathbf{n}_k(t_i))^2 . \tag{8}$$

4 Solving for $\{\mathbf{e}_{\mathbf{c}_k^2}\}$ and $\{\mathbf{e}_{\theta_k}\}$

This section gives details about how we solve for the geometry $\{\mathbf{e}_{\mathbf{c}_k^2}\}$ and normals $\{\mathbf{e}_{\theta_k}\}$ of K conjugate curve pairs, assuming the relative position (d_x) of the driver surface and the number (K) of conjugate curves are given.

we find that there are too many high-level constraints in our optimziation problem, so it is too difficult to solve. Especially, the gradient of the form-closure condition 2 cannot be calculated.

Therefore, our idea to first generate a variety of multi-point conjugation joint candidates consisting of K conjugate curve pairs by a gradient-based optimization (we take mpcMech_3R as an example):

$$\begin{aligned} \min_{\{\mathbf{e}_{\mathbf{c}_k^2}\}, \{\mathbf{e}_{\theta_k}\}} \quad & E_{\text{k}} + \omega_5 E_{\text{vary}}, \\ \text{s.t.} \quad & C_{\text{closure\_3R}} \leqslant 0, \\ & L_1 \leqslant \text{Length}(\mathbf{c}_k^2(t)) \leqslant L_2, \\ & z_k^{\min} \leqslant \mathbf{c}_k^2(t).z \leqslant z_k^{\max} \end{aligned} \tag{9}$$

where

$$E_{\text{vary}} = \tilde{\mathbf{n}}_k(t_v) \cdot \tilde{\mathbf{n}}_k(t_v + T/2). \tag{10}$$

E_{vary} helps to generate many conjugate curve pairs with different geometry by using different t_v and $\omega_6 = 20.0$ in our experiments. Other expressions is mentioned in Section 5.1 and 5.2 in the paper. Note that $\tilde{\mathbf{n}}_k$ for mpcMech_1R is +1 or −1, so we discard E_{vary} in this case. And for mpcMech_1R or mpcMech_1R1T, we remove the constraint $C_{\text{closure\_3R}}$ and add the corresponding term $E_{\text{closure\_1R}}$ or $E_{\text{closure\_1R1T}}$ to the objective function. And the coefficient ω_6 of the $E_{\text{closure\_1R}}$ or $E_{\text{closure\_1R1T}}$ is set 0.1 in our experiments.

To generate multi-point conjugation joint candidates, we take M uniform distributed time sequence $\{t_{vj}\}_{1 \leqslant j \leqslant M}$ in a whole motion period. For each t_{vj}, we run the above optimization for each $k \in [1, K]$ to obtain a multi-point conjugation joint candidate consisting of K conjugate curve pairs . By this, we obtain M multi-point conjugation joint candidates. In our experiments, we set $M = 30$.

Thanks to the term E_{vary}, the conjugate curve pairs in the candidates are so diverse, but we can not guarantee that there exists a candidate satisfying the dynamic form closure condition. We observe that due to the constraint for the z-coordinate of $\mathbf{c}_k^2$, two multi-point conjugation joint candidates can exchange the kth $(1 \leqslant k \leqslant K)$ conjugate curve pair to generate a new candidate. So we can enumerate all the combinations of these M candidates and find the best combination that minimizing E_{valiTime}. However, enumeration is time-consuming, so we use the genetic algorithm to find a combination of the multi-point conjugation joint candidates to satisfy the dynamic form closure condition. The fitness function of the genetic algorithm is based on the two objective functions in the paper. In detail, we consider an individual $\mathbf{I}_i$ is better than another one $\mathbf{I}_j$ if one of the following condition is satisfied:

1. $E_{\text{valiTime}}(\mathbf{I}_i) > E_{\text{valiTime}}(\mathbf{I}_j)$
2. $E_{\text{valiTime}}(\mathbf{I}_i) = E_{\text{valiTime}}(\mathbf{I}_j) \& E_{\text{maxDist}}(\mathbf{I}_i) < E_{\text{maxDist}}(\mathbf{I}_j)$

In the genetic algorithm we used, we select very common operators: truncation selection, one point crossover and bit flip mutation. We perform the above three operations N_I times, and select the best individual $\mathbf{I}^{optim}$ in the final population according to the fitness function. In our experiments, we set $N_I = 300$. If $E_{\text{valiTime}} < 1.0$ for $\mathbf{I}^{optim}$, we increase the number of the conjugate curve pairs to $K+1$ and repeat the above process until we find a solution that satisfy $E_{\text{valiTime}} = 1.0$.