Supplementary Material for *Computational Design of Coordinate-Motion Assemblies*

This supplementary includes 2 sections. Section 1 presents the theoretical analysis of the relation between contact geometry and disassembly motion. Section 2 provides the proofs of the proposition of unique disassembly conditions in the main paper.

1 RELATION BETWEEN CONTACT GEOMETRY AND DISASSEMBLY MOTION

While the main paper focuses on unique disassembly, our theoretical analysis extends to general disassembly motions. We derive a proposition for the relation between contact geometry and general disassembly motion, and from this, Proposition 3.3 in the main paper is a special case for unique disassembly motion.

1.1 General contact classification

Consider a general assembly that has disassembly motion set $V \subset \mathbb{R}^n$. We define *sliding contacts* as those that remain active (maintain contact) for any motion $\mathbf{Y} \in V$; the separating contacts as the contacts that must separate for one disassembly motion $\mathbf{Y} \in V$; and the blocking contacts as the contacts that block at least one motion in $\mathbf{Y} \in V$.

Definition 1.1 (General contact classification). Let set $V \subset \mathbb{R}^n$, matrix $\mathbf{B} \in M_{m\times n}(\mathbb{R})$. Denote $\mathbf{b}_i$ as i-th row vector, $\mathbf{B} = (\mathbf{b}_1^\top, \dots, \mathbf{b}_m^\top)^\top$. Classify row indices of matrix $\mathbf{B}$ based on V:

- $I_{\text{sld}} = \{1 \le i \le m | \mathbf{b}_i\mathbf{Y} = 0, \forall\, \mathbf{Y} \in V\}$;
- $I_{\text{spt}} = \{1 \le i \le m | \mathbf{b}_i\mathbf{Y} \ge 0, \forall\, \mathbf{Y} \in V, \text{and } \exists\, \hat{\mathbf{Y}} \in V, \mathbf{b}_i\hat{\mathbf{Y}} > 0\}$;
- $I_{\text{blk}} = \{1 \le i \le m | \exists\, \hat{\mathbf{Y}} \in V, \mathbf{b}_i\hat{\mathbf{Y}} < 0\}$.

We call $\{I_{\text{sld}}, I_{\text{spt}}, I_{\text{blk}}\}$ the contact classification of motion set V.

The classification is well-defined because the three index sets are pairwise disjoint, $I_{\text{sld}} \cap I_{\text{spt}} = \varnothing$, $I_{\text{sld}} \cap I_{\text{blk}} = \varnothing$, $I_{\text{spt}} \cap I_{\text{blk}} = \varnothing$, and $I_{\text{sld}} \cup I_{\text{spt}} \cup I_{\text{blk}} = \{1, 2, \dots, m\}$. Note that the set V can be arbitrarily chosen in the definition above. We now seek the conditions on the contact normals $\{\mathbf{b}_i\}$ such that V becomes exactly the solution set of $\mathbf{BY} \ge \mathbf{0}$. This leads to the following proposition:

Proposition 1.2. *Let nonempty set $V \subset \mathbb{R}^n$, matrix $\mathbf{B} \in M_{m\times n}(\mathbb{R})$, and $\{I_{sld}, I_{spt}, I_{blk}\}$ be the contact classifications of set V. The following two conditions are equivalent:*

1. *$V = \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{BY} \ge \mathbf{0}\}$;*
2. *The conditions on $\{\mathbf{b}_i\}_{I_{sld}}$, $\{\mathbf{b}_i\}_{I_{spt}}$, and $\{\mathbf{b}_i\}_{I_{blk}}$ are as follows:*
   - (a) *$\forall\, \mathbf{v} \in V^\perp \setminus \{\mathbf{0}\}$, $\exists\, i \in I_{sld}$ such that $\mathbf{b}_i\mathbf{v} < 0$;*
   - (b) *Consider the following two cases:*
     - *Case 1: $span(V) = V$. Then $I_{spt} = \varnothing$.*
     - *Case 2: $span(V) \ne V$. Then $I_{spt} \ne \varnothing$, and moreover $\forall\, \mathbf{v} \in span(V) \setminus V$, $\exists\, i \in I_{spt}$ such that $\mathbf{b}_i\mathbf{v} < 0$;*
   - (c) *$I_{blk} = \varnothing$,*

where $V^\perp = \{\mathbf{v} \in \mathbb{R}^n | \forall \mathbf{y} \in V, \mathbf{y}^\top\mathbf{v} = 0\}$ is the orthogonal complement of V, and $span(V) = \{\sum_{i=1}^{k} \lambda_i\mathbf{y}_i | k \in \mathbb{Z}_{>0}, \lambda_i \in \mathbb{R}, \mathbf{y}_i \in V\}$ is the linear span of V.

2 PROOF OF PROPOSITIONS

2.1 Proof of Proposition 1.2

Proof.

- Proof of (1) $\Rightarrow$ (2):
  - (i) Because $\mathbf{b}_i\mathbf{Y} \ge \mathbf{0}$, $\forall 1 \le i \le m$, condition (c) holds.
  - (ii) Proof of condition (b):
    - Case 1: $\text{span}(V) = V$. Because $V = \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{BY} \ge \mathbf{0}\}$, $\forall\, \mathbf{y} \in V$, $\mathbf{b}_i\mathbf{y} \ge 0$, $\forall i \in \{1, \dots, m\}$. Because $V = \text{span}(V)$, $-\mathbf{y} \in V$ as well, we have $\mathbf{b}_i(-\mathbf{y}) \ge 0$, $\forall i \in \{1, \dots, m\}$. So $\mathbf{b}_i\mathbf{y} = 0$, $\forall i \in \{1, \dots, m\}$. This means that $I_{\text{spt}} = \varnothing$.
    - Case 2: $\text{span}(V) \ne V$. We claim that $\exists\, \mathbf{y} \in V$, s.t. $\mathbf{b}_{i^*}\mathbf{y} > 0$, for some $i^* \in \{1, \dots, m\}$. Otherwise, if no such $\mathbf{y}$ exists, then $I_{\text{sld}} = \{1, \dots, m\}$, and thus $V = \{\mathbf{Y} \in \mathbb{R}^n \mid \mathbf{BY} = \mathbf{0}\}$. This contradicts with $\text{span}(V) \ne V$. So $i^* \in I_{\text{spt}}$, and thus $I_{\text{spt}} \ne \varnothing$. $\forall\, \mathbf{v} \in \text{span}(V) \setminus V$, we have $\forall\, i \in I_{\text{sld}}$, $\mathbf{b}_i\mathbf{v} = 0$. Because $\mathbf{v} \notin V$, $\exists\, i^{**} \in I_{\text{spt}} \cup I_{\text{blk}}$, s.t. $\mathbf{b}_{i^{**}}\mathbf{v} < 0$. Because $I_{\text{blk}} = \varnothing$ as proved above, $i^{**} \in I_{\text{spt}}$.
  - (iii) To prove that condition (a) holds, we prove by contradiction: Assume there exists $\mathbf{v} \in V^\perp \setminus \{\mathbf{0}\}$ such that $\forall i \in I_{\text{sld}}$, $\mathbf{b}_i\mathbf{v} \ge 0$. We consider two cases:
    - If $I_{\text{spt}} = \varnothing$, then I_{sld} contains all indices (since $I_{\text{blk}} = \varnothing$). Thus $\mathbf{Bv} \ge \mathbf{0}$, which implies $\mathbf{v} \in V$. However, $\mathbf{v} \in V^\perp$ implies $\mathbf{v} \cdot \mathbf{v} = 0$, so $\mathbf{v} = \mathbf{0}$, which contradicts the assumption $\mathbf{v} \ne \mathbf{0}$.
    - If $I_{\text{spt}} \ne \varnothing$, then for each $i \in I_{\text{spt}}$, there exists $\mathbf{Y}_i \in V$ such that $\mathbf{b}_i\mathbf{Y}_i > 0$. Let $\mathbf{v}_i = \mathbf{v} + k_i\mathbf{Y}_i$, where $k_i > 0$ is a sufficiently large constant such that $\mathbf{b}_i\mathbf{v}_i > 0$. Define the vector:
      $$\tilde{\mathbf{v}} = \mathbf{v} + \sum_{j \in I_{\text{spt}}} k_j\mathbf{Y}_j. \tag{1}$$
      We verify the constraints for $\tilde{\mathbf{v}}$: For $i \in I_{\text{sld}}$, $\mathbf{b}_i\tilde{\mathbf{v}} = \mathbf{b}_i\mathbf{v} + 0 \ge 0$ (by assumption). For $i \in I_{\text{spt}}$, $\mathbf{b}_i\tilde{\mathbf{v}} = \mathbf{b}_i\mathbf{v} + \sum_{j \in I_{\text{spt}}} k_j\mathbf{b}_i\mathbf{Y}_j$. Since $\mathbf{b}_i\mathbf{Y}_j \ge 0$ and $k_i\mathbf{b}_i\mathbf{Y}_i > 0$, we can choose k's large enough such that $\mathbf{b}_i\tilde{\mathbf{v}} > 0$. Thus $\mathbf{B}\tilde{\mathbf{v}} \ge \mathbf{0}$, which implies $\tilde{\mathbf{v}} \in V$.
      Now consider the dot product $\tilde{\mathbf{v}} \cdot \mathbf{v}$:
      $$\tilde{\mathbf{v}} \cdot \mathbf{v} = \|\mathbf{v}\|^2 + \sum_{j \in I_{\text{spt}}} k_j(\mathbf{Y}_j \cdot \mathbf{v}). \tag{2}$$
      Since $\mathbf{v} \in V^\perp$ and $\mathbf{Y}_j \in V$, we have $\mathbf{Y}_j \cdot \mathbf{v} = 0$. Therefore, $\tilde{\mathbf{v}} \cdot \mathbf{v} = \|\mathbf{v}\|^2 > 0$. However, since $\tilde{\mathbf{v}} \in V$ and $\mathbf{v} \in V^\perp$, their dot product must be zero. This is a contradiction.

    Thus, condition (a) must hold.
- Proof of (2) $\Rightarrow$ (1): $\forall\, \mathbf{v} \in V$, because $I_{\text{blk}} = \varnothing$, we have $\mathbf{Bv} \ge \mathbf{0}$. So $V \subset \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{BY} \ge \mathbf{0}\}$. Next we prove $V^c \subset \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{BY} \ge \mathbf{0}\}^c$.
  $\forall\, \mathbf{v} \in V^c$, note that $\text{span}(V)$ is a subspace of $\mathbb{R}^n$, so we have the orthogonal decomposition $\mathbb{R}^n = (\text{span}(V))^\perp \oplus \text{span}(V)$. Hence

$\mathbf{v}$ admits an orthogonal decomposition $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in (\mathrm{span}(V))^{\perp}$ and $\mathbf{v}_2 \in \mathrm{span}(V)$. We discuss two cases:

- If $\mathbf{v}_1 \neq \mathbf{0}$, from condition (a), $\exists i^* \in I_{\mathrm{sld}}$, s.t. $\mathbf{b}_{i^*}\mathbf{v}_1 < 0$. Because $\mathbf{b}_{i^*}\mathbf{v}_2 = 0$ by the definition of I_{sld}, we have $\mathbf{b}_{i^*}\mathbf{v} = \mathbf{b}_{i^*}\mathbf{v}_1 < 0$. So $\mathbf{v} \notin \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}\}$.
- If $\mathbf{v}_1 = \mathbf{0}$, then $\mathbf{v} \in \mathrm{span}(V) \setminus V$. From condition (b), $\exists i^{**} \in I_{\mathrm{spt}}$, s.t. $\mathbf{b}_{i^{**}}\mathbf{v} < 0$. So $\mathbf{v} \notin \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}\}$.

In both cases, we have $\mathbf{v} \notin \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}\}$, so $V^c \subset \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}\}^c$.

Above all, we have $V = \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}\}$.

□

2.2 Proof of Unique Disassembly Conditions

Based on Proposition 1.2, we can prove Proposition 3.3 in the main paper, which we restate here for clarity.

Proposition 2.1 (Unique disassembly conditions). *Let vector $\mathbf{Y}^* \in \mathbb{R}^n$, matrix $\mathbf{B} = (\mathbf{b}_1^{\mathsf{T}}, \ldots, \mathbf{b}_m^{\mathsf{T}})^{\mathsf{T}} \in M_{m\times n}(\mathbb{R})$, set $W = \{\mathbf{Y} \in \mathbb{R}^n | \mathbf{B}\mathbf{Y} \geq \mathbf{0}, \mathbf{Y} \neq \mathbf{0}\}$, and $\{I_{sld}, I_{spt}, I_{blk}\}$ is the contact classifications of $\mathbf{Y}^*$. The following two statements are equivalent:*

- *(1) $W = \{\lambda \mathbf{Y}^* | \lambda > 0\}$;*
- *(2) The following conditions hold:*
  - *(a) $\forall\, \mathbf{v} \in \mathbf{Y}^{*\perp} \setminus \{\mathbf{0}\}$, $\exists j \in I_{sld}$, $\mathbf{b}_j\mathbf{v} < 0$ (sliding condition);*
  - *(b) $I_{spt} \neq \varnothing$ (separating condition);*
  - *(c) $I_{blk} = \varnothing$ (blocking condition).*

Proof. Let $V = \{\lambda \mathbf{Y}^* | \lambda \geq 0\}$, then contact classification $\{I_{\mathrm{sld}}, I_{\mathrm{spt}}, I_{\mathrm{blk}}\}$ of V is the same as that of $\mathbf{Y}^*$. Condtion (1) and condition (2) in Proposition 2.1 are equivalent to condition (1) and condition (2) in Proposition 1.2, respectively. This proposition thus follows directly from Proposition 1.2.

□